Chapter 6: Matrices

6.1 Matrix Operations

1. Given the matrices
   \[ A = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -1 & 1 & 0 \\ 4 & -2 & 2 \end{pmatrix}, \]
   compute the following, if they are defined.
   
   a. AB
   b. BA
   c. A + B
   d. 3A

6.2 Solving Systems of Linear Equations

2. Find all solutions to the given system of equations using either Gaussian elimination or Gauss-Jordan elimination.

   \[
   \begin{align*}
   4x + 3y &= 5 \\
   3x - 2y &= 8
   \end{align*}
   \]

3. Find all solutions to the following system of equations.

   \[
   \begin{align*}
   x + 2z &= 1 \\
   x + y + z &= 3 \\
   2x + 3y + 2z &= 9
   \end{align*}
   \]

6.3 Finding a Matrix Inverse and Determinant

4. Find the determinants of the following matrices A and B.

   \[
   A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & -1 & -2 \\ -3 & 3 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 4 & 1 & 0 \\ 3 & 2 & 1 \\ 0 & 4 & 2 \end{pmatrix}
   \]

5. Find the inverses of the following matrices.

   \[
   A = \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}
   \]

6.4 Computing Eigenvalues and Eigenvectors

6. Compute the eigenvalues and eigenvectors for the following matrix.

   \[
   A = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}
   \]
6.5 Solving Difference Equations

7. Solve the following initial value difference equation. (Note: Not all classes cover this material.)

\[ x_{n+1} - 5x_n + 6x_{n-1} = 0 \quad x_0 = 4 \quad x_1 = 9 \]

Chapter 7: Functions of Several Variables

7.1 Functions of Several Variables

8. The surface area of a person (in m²) can be approximated using a formula known as the Haycock formula. The formula is a function of two variables, the person’s height in centimeters and the person’s weight in kilograms. The formula is as follows

\[ S(h, w) = 0.024265h^{0.3964}w^{0.5378} \]

Estimate the surface area of a person who is 165 cm tall and weighs 80 kg.

7.2 Partial Derivatives

9. Find first and second order partial derivatives of the function \( f(x, y) = 3xy^2 + 2xy + x^2 \).

7.3 Maximum-Minimum Problems

10. Find all relative maximum and minimum values of the function and verify your result using the \( D \)-Test.

\[ f(x, y) = 4xy - x^3 - y^2 \]

Chapter 8: First Order Differential Equations

8.2 Linear First-Order Differential Equations

11. Find the general solutions to the given differential equations.

a. \( y' + 2y = e^t \)

b. \( y' + \frac{2}{x}y = \frac{2e^{x^2}}{x} \)

c. \( y' + y \tan x = \cos x \)

12. Solve the following the initial value problem.

\[ \frac{dy}{dx} = x - y, \quad y(0) = 2 \]

8.3 Autonomous Differential Equations and Stability

13. Many bacteria strains are used by the dairy industry to produce different types of fermented milks and yogurts. During a fermentation experiment, the population \( y \) (in millions per mL) of bacteria *Lactobacillus fermentum* after \( t \) hours in a wheat medium satisfied the differential equation

\[ y' = 0.532y(1 - \frac{y}{1900}) \]

a. Determine the equilibrium solutions to the differential equation.

b. Assess the stability of each equilibrium solution.

c. Initially, 8 million bacteria per mL were present. Determine the number of bacteria present after 5 hr.
8.4 Separable Differential Equations

14. Find the general solution to the given separable differential equations

   a. \( \frac{dy}{dx} = y \tan x \)
   b. \( \frac{1}{\sin x} \frac{dy}{dx} = y \cos x \)
   c. \( \frac{1}{(\sin x + \cos x)^2} \frac{dy}{dx} = y^2 \)

Chapter 9: Higher-Order and Systems of Differential Equations

9.1 Higher-Order Homogeneous Differential Equations

15. Solve for the general solution of the given homogeneous differential equations.

   a. \( y''' + y'' - 6y' = 0 \)
   b. \( y'' + 4y' + 4y = 0 \)
   c. \( y'' + 2y' + 2y = 0 \)

9.2 Higher-Order Nonhomogeneous Differential Equations

16. Find the solution to the given differential equation

   \( y'' + 4y = 16x \)

   with initial conditions \( y(0) = 2 \) and \( y'(0) = -3 \).

9.3 Systems of Linear Differential Equations

17. Solve the initial value problem

   \[
   \begin{align*}
   x' &= 2x + y + 3 \\
y' &= 5x - 2y + 12
   \end{align*}
   \]

   given the initial conditions \( x(0) = 6 \) and \( y(0) = -3 \).

9.4 Matrices and Trajectories

18. Use the reduction method to find the general solution of the system of differential equations.

   \[
   \begin{align*}
   x' &= y + 2t + 3 \\
y' &= -x + 4t - 2
   \end{align*}
   \]

19. Use matrix methods to find the general solution and the stability of the origin for the system of differential equations. Sketch the phase portrait.

   \[
   \begin{align*}
   x' &= -x + y \\
y' &= 2y
   \end{align*}
   \]

20. Find the general solution to the given systems of differential equations.

   a. \( \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \)
   a. \( \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \)
9.5 Models of Population Biology

21. On a wildlife preserve, the population of gorillas is modeled by the differential equation

\[ \frac{dy}{dt} = 0.06y(60 - y) \]

Where \( t \) is measured in years.

a. Find the equilibrium solutions to the differential equation.

b. Compute the general solution to the differential equation.

c. Given that the preserve had 45 gorillas 4 years ago, find the number of gorillas currently on the preserve.

d. Describe what happens to the gorilla population in the long run.
Answers

1. a. \( \mathbf{AB} = \begin{pmatrix} 2 & 0 & 2 \\ 13 & -7 & 6 \end{pmatrix} \)

b. \( \mathbf{BA} \) is not defined.

c. \( \mathbf{A} + \mathbf{B} \) is also not defined.

d. \( 3\mathbf{A} = \begin{pmatrix} 6 & 3 \\ -3 & 9 \end{pmatrix} \)

2. \( x = 2, \ y = -1 \).

3. The only solution is \( x = -1, \ y = 3 \) and \( z = 1 \).

4. \( \det \mathbf{A} = 1, \ \det \mathbf{B} = -6 \)

5. \( \mathbf{A}^{-1} = \begin{pmatrix} -3 & 2 \\ \frac{5}{2} & -\frac{3}{2} \end{pmatrix}, \ \mathbf{B}^{-1} = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ 1 & 0 \end{pmatrix} \)

6. \( \lambda_1 = 2 \ \ \lambda_2 = 3 \)

\( \mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \ \ \mathbf{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \)

7. \( x_n = 3(2^n) + 3^n \)

8. \( 1.9385 \text{ m}^2 \)

9. \( f_x(x, y) = 3y^2 + 2y + 2x \)
\( f_y(x, y) = 6xy + 2x \)
\( f_{xx}(x, y) = 2 \)
\( f_{xy}(x, y) = 6y + 2 \)
\( f_{yx}(x, y) = 6y + 2 \)
\( f_{yy}(x, y) = 6x \)

10. \((0, 0)\) is a saddle point and \( (\frac{8}{3}, \frac{16}{3}) \) is a local maximum.

11. a. \( y_g = Ce^{-2t} + \frac{1}{3}e^t \)

b. \( y_g = \frac{e^{x^2}}{x^2} + \frac{C}{x^2} \)

c. \( y_g = x|\cos x| + C|\cos x| \)

12. \( y(x) = 3e^{-x} + x - 1 \)

13. a. \( y(t) = 0 \) and \( y(t) = 1900 \).

b. \( y(t) = 0 \) is unstable and \( y(t) = P \) is stable.

c. \( 108.3 \text{ million bacteria per mL} \).

14. a. \( y_g = C_1|\sec x| \)

b. \( y_g = C_1e^\frac{1}{4}\sin^2 x \)

c. \( y_g = \frac{-1}{x + \sin^2 x + C} \)
15. a. \( y_g(t) = C_1 + C_2 e^{2t} + C_3 e^{-3t} \)
   b. \( y_g(t) = C_1 e^{-2t} + C_2 t e^{-2t} \)
   c. \( y_g(t) = C_1 e^{\alpha t} \sin \beta t + C_2 e^{\alpha t} \cos \beta t \)

16. \( y(x) = -\frac{7}{2} \sin 2x + 2 \cos 2x + 4x \)

17. \( y(t) = \frac{20}{3} e^{3t} + \frac{10}{3} e^{-3t} - 3 \)

18. \( x(t) = C_1 \sin t + C_2 \cos t + 4t \)
   \( y(t) = C_1 \cos t - C_2 \sin t - 2t + 1 \)

19. \( x(t) = C_1 e^{-t} + C_2 e^{2t} \)
   \( y(t) = 3C_2 e^{2t} \)

20. a. \( x(t) = x_0 e^{2t} + y_0 t e^{2t} \)
    \( y(t) = y_0 e^{2t} \)
   b. \( \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = C_1 \begin{pmatrix} e^{2t} \cos t - e^{2t} \sin t \\ -e^{2t} \cos t + e^{2t} \sin t \end{pmatrix} + C_2 \begin{pmatrix} -e^{2t} \sin t \\ e^{2t} \cos t + e^{2t} \sin t \end{pmatrix} \)

21. a. \( y(t) = 0 \) and \( y(t) = 60 \)
   b. \( y_g = \frac{60}{1 + C_2 e^{-3.6t}} \)
   c. 60 gorillas
   d. \( y(t) = 60 \) is a stable solution, so if there are any gorillas on the reserve, the population will tend towards 60 gorillas in the long run.