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1 Geometry of $\mathbb{R}^2$

1.1 Conic Sections

A cross-section of a parabolic reflector is shown in the figure. The bulb is located at the focus and the opening at the focus is 10 cm.

(a) Find an equation of the parabola.

(b) Find the diameter of the opening $|CD|$, 11 cm from the vertex.

(c) Parametrize the bottom half of the parabola.

1.2 Parametric Equations

By first eliminating the parameter, describe the motion of a particle following the graph of the parametric equations:

$$x = 2 \sin t, \quad y = 4 + \cos t, \quad 0 \leq t \leq \frac{3\pi}{2}$$

1.3 More Parametric Equations

Find the Cartesian equations for each of the curves described by the parametric equations below:

(a) $x = e^t - 1, \quad y = e^{2t}$

(b) $x = \tan^2 \theta, \quad y = \sec \theta, \quad \frac{-\pi}{2} < \theta < \frac{\pi}{2}$

1.4 Polar Coordinates

Sketch the curve

$$(x^2 + y^2)^3 = 4x^2y^2$$

2 Complex Numbers

2.1 Polar Form

Use Euler’s formula to prove that:

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \quad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$
2.2 Roots of Unity
Find all the cube roots of 1 and show that if one of the complex roots (i.e. nonzero imaginary part) is labeled $z$ then the other complex root is $z^2$.

3 Geometry of $\mathbb{R}^3$

3.1 The Distance Formula
Describe the set of points $P$ such that the distance from $P$ to $A(-1, 5, 3)$ is equal to the distance from $P$ to $B(6, 2, -2)$.

3.2 Dot Product and Cross Product
For the two vectors $\mathbf{A}, \mathbf{B}$ in $\mathbb{R}^3$:
(a) Find the value of $(|\mathbf{A} \times \mathbf{B}|)^2 + (\mathbf{A} \cdot \mathbf{B})^2$.
(b) Show that if $\mathbf{A} - \mathbf{B}$ and $\mathbf{A} + \mathbf{B}$ are orthogonal, then $\mathbf{A}$ and $\mathbf{B}$ must have the same length.

3.3 Lines in $\mathbb{R}^3$
Determine whether the following lines are parallel, intersect, or are skew. If they intersect, find the point of intersection.

\[ l_1 : \frac{x - 2}{1} = \frac{y - 3}{-2} = \frac{z - 1}{-3} \]
\[ l_2 : \frac{x - 3}{1} = \frac{y + 4}{3} = \frac{z - 2}{-7} \]

3.4 Distances Between Lines
Find the distance between the lines $\mathbf{r}_1 = (1 + t)i - 2j - tk$ and $\mathbf{r}_2 = -si + (2 + s)j - k$.

3.5 Equations of Planes
Find the equation of the plane with $x$-intercept $a$, $y$-intercept $b$, and $z$-intercept $c$.

3.6 More Equations of Planes
Find the equation of the plane that contains the line of intersection of the planes $x - z = 1$ and $y + 2z = 3$ and is perpendicular to the plane $x + y - 2z = 1$.

3.7 Parametric Equations of Curves
Two particles travel along the following space curves:
\[ \mathbf{r}_1(t) = \langle t, t^2, t^3 \rangle, \quad \mathbf{r}_2(t) = \langle 1 + 2t, 1 + 6t, 1 + 14t \rangle \]
Do the particles collide? Do their paths cross?

3.8 Sketching Quadric Surfaces
Use traces to sketch and identify the surface with the following equation:
\[ 4x^2 - 16y^2 + z^2 = 16 \]
3.9 Surfaces of Rotations
Given the line \( z = y \tan \theta \) in the \( yz \)-plane, where \( 0 < \theta < \pi/2 \), find the equation of the surface generated by rotating the line about the \( z \)-axis.

3.10 Equations of Quadric Surfaces
Find an equation for the surface consisting of all points \( P \) for which the distance from \( P \) to the \( x \)-axis is twice the distance from \( P \) to the \( yz \)-plane. Identify the surface.

3.11 Contours
Draw a contour map of the following function showing several level curves:

\[
f(x, y) = x^3 - y
\]

3.12 Points in Cylindrical Coordinates
Change the following points to the specified coordinate system:

(a) \((2\sqrt{3}, 2, -1)\) from rectangular coordinates to cylindrical coordinates

(b) \((1, 1, 1)\) from cylindrical coordinates to rectangular coordinates

3.13 Points in Spherical Coordinates
Change the following points to the specified coordinate system:

(a) \((4, -\frac{\pi}{4}, \frac{\pi}{3})\) from spherical coordinates to rectangular coordinates

(b) \((-1, 1, \sqrt{2})\) from rectangular coordinates to spherical coordinates

3.14 Parametric Equations of Surfaces
Find a parametric representation for the following surfaces:

(a) The part of the sphere \( x^2 + y^2 + z^2 = 4 \) that lies above the cone \( z = \sqrt{x^2 + y^2} \)

(b) The part of the ellipsoid \( x^2 + 2y^2 + 3z^2 = 1 \) that lies to the left of the \( xz \)-plane

4 Linear Algebra

4.1 Systems of Equations
Solve the following system of equation using substitution and/or elimination:

\[
\begin{align*}
2x - y + z &= 3 \\
3x + 2y - z &= -1 \\
x - 3y + 2z &= 2
\end{align*}
\]

4.2 More on Systems of Equations
Solve the system of equations from the previous problem using Gaussian elimination. Interpret your solution.
4.3 Reduced Row-Echelon
Given that a system of 3 linear equations, each of 3 variables, has the following solution, construct the reduced row-echelon matrix the solution came from.

\[ x = 1 - 4t, \quad y = 3 + 3t, \quad z = 2 - t \]

4.4 Determinants
Calculate the determinant of the following matrix.

\[
\begin{pmatrix}
3 & 1 & 3 & 6 \\
4 & -7 & -3 & 5 \\
1 & 3 & 4 & -3 \\
3 & 0 & 2 & 7
\end{pmatrix}
\]

4.5 Inverses and Transpositions
A matrix \( A \) is called orthogonal if \( AA^T = I \), where \( I \) is the identity matrix (i.e. \( A^{-1} = A^T \)). Given the following matrix, compute the inverse using (a) Row reduction, and (b) using \( 2 \times 2 \) inverse formula. Verify that \( A \) is in fact orthogonal.

\[
A = \begin{pmatrix}
-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{pmatrix}
\]

4.6 Products of Determinants
For any two \( 2 \times 2 \) matrices \( A \) and \( B \), show that \( \det(AB) = \det(A)\det(B) \).

4.7 Matrix Arithmetic and Characteristic Equations
The Caley-Hamilton theorem states that "A matrix satisfies its own characteristic equation." Verify this theorem for the following matrix

\[
M = \begin{pmatrix}
5 & -2 \\
-2 & 2
\end{pmatrix}
\]

4.8 Eigenvalues and Eigenvectors
Find the eigenvalues of the real symmetric matrix

\[
\begin{pmatrix}
A & H \\
H & B
\end{pmatrix}
\]

Where \( A, H, B \in \mathbb{R} \). Show that the eigenvalues are real. In the case where \( A = 4, B = 1, \) and \( H = 2 \), show that the eigenvectors are orthogonal.
5 Geometry of $\mathbb{R}^2$ - Solutions

1.1 Conic Sections

(a) $y^2 = 10x$

(b) $|CD| = 2\sqrt{110}$

(c) $x = t, \quad y = -\sqrt{10t}$

1.2 Parametric Equations

The particle travels from point A to point C through the point B ($t = \frac{\pi}{2}$)

1.3 More Parametric Equations

(a) $y = (x + 1)^2$

(b) $y = \sqrt{1 + x}$

1.4 Polar Coordinates

The sketch is as shown (traced twice).
6  Complex Numbers - Solutions

2.1  Polar Form

\[ e^{ix} + e^{-ix} = 2 \cos x \implies \cos x = \frac{e^{ix} + e^{-ix}}{2} \]

\[ e^{ix} - e^{-ix} = 2i \sin x \implies \sin x = \frac{e^{ix} - e^{-ix}}{2i} \]

2.2  Roots of Unity

\[ \omega_0 = 1 \]
\[ \omega_1 = \frac{1}{2} + \frac{\sqrt{3}}{2}i \]
\[ \omega_2 = \frac{1}{2} - \frac{\sqrt{3}}{2}i \]
\[ (\omega_1)^2 = \omega_2 \]
\[ (\omega_2)^2 = \omega_1 \]

7  Geometry of \( \mathbb{R}^3 \) - Solutions

3.1  The Distance Formula

\[ 14x - 6y - 10z = 9 \]

The set of points \( P \) is a plane in \( \mathbb{R}^3 \) with a normal vector \( \mathbf{n} = < 7, -3, -5 > \) going through the point \( C\left( \frac{5}{2}, \frac{7}{2}, \frac{1}{2} \right) \).

3.2  Dot Product and Cross Product

(a) \( (\| \mathbf{A} \times \mathbf{B} \|)^2 + (\mathbf{A} \cdot \mathbf{B})^2 = \| \mathbf{A} \|^2 \| \mathbf{B} \|^2 \)

(b) \( \mathbf{A} - \mathbf{B} \cdot (\mathbf{A} + \mathbf{B}) = 0 \implies \| \mathbf{A} \| = \| \mathbf{B} \| \)

3.3  Lines in \( \mathbb{R}^3 \)

The lines intersect. The point of intersection is \( (4, -1, -5) \)

3.4  Distances Between Lines

\[ D = \frac{2}{\sqrt{3}} \]

3.5  Equations of Planes

\[ \frac{1}{a} x + \frac{1}{b} y + \frac{1}{c} z = 1 \]
3.6 More Equations of Planes

\[ x + y + z = 4 \]

3.7 Parametric Equations of Curves

The particles do not collide. The intersection of path occurs at \( \mathbf{r}_1(1) = (1, 1, 1), \mathbf{r}_1(2) = (2, 4, 8) \)

3.8 Sketching Quadric Surfaces

The surface at hand is hyperboloid of one sheet.
3.9 Surfaces of Rotations

\[ x^2 + y^2 = \frac{z^2}{\tan^2 \theta} \]

3.10 Equations of Quadric Surfaces

\[ y^2 + z^2 = 4x^2 \]

We can easily identify the equation as the equation of a cone (one can also use traces to identify this surface)

3.11 Contours

3.12 Points in Cylindrical Coordinates

(a) \((4, \frac{\pi}{6}, -1)\)

(b) \((\cos 1, \sin 1, 1) \approx (0.54030, 0.84147, 1)\)
3.13 Points in Spherical Coordinates
(a) \((\sqrt{6}, -\sqrt{6}, 2)\)
(b) \((2, \frac{3\pi}{4}, \frac{\pi}{4})\)

3.14 Parametric Equations of Surfaces
(a) \(x = 2\cos\theta\sin\phi, \quad y = 2\sin\theta\sin\phi, \quad z = 2\cos\phi\)
(b) \(x = u, \quad y = -\sqrt{\frac{1}{2}(1 - u^2 - 3v^2)}, \quad z = v\)

8 Linear Algebra - Solutions

4.1 Systems of Equations
\[
\begin{align*}
x &= -1 \\
y &= 7 \\
z &= 12
\end{align*}
\]

4.2 More on Systems of Equations
\[
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix} =
\begin{pmatrix}
-1 \\
7 \\
12
\end{pmatrix}
\]
The results tell us that all three planes we were given the equations of intersect at a single point and the point is \((x, y, z) = (-1, 7, 12)\)

4.3 Reduced Row-Echelon
\[
\begin{pmatrix}
1 & 0 & -4 & -7 \\
0 & 1 & 3 & 9 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

4.4 Determinants
\[
\det(A) = -2
\]

4.5 Inverses and Transpositions
\[
A^{-1} =
\begin{pmatrix}
-\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}
\end{pmatrix}
\]
\[
AA^T = AA^{-1} = I
\]
4.6 Products of Determinants

\[
\begin{align*}
\det(A) \det(B) &= ada'd' - adb'c' - bca'd' + bcb'c' \\
\det(AB) &= ada'd' + bcb'c' - bca'd' - adb'c'
\end{align*}
\]

4.7 Matrix Arithmetic and Characteristic Equations

\[
\lambda^2 - 7\lambda + 6 = 0
\]

\[
\begin{pmatrix}
5 & -2 \\
-2 & 2
\end{pmatrix}^2 - 7 \begin{pmatrix}
5 & -2 \\
-2 & 2
\end{pmatrix} + 6 \begin{pmatrix}
1 & 0 \\
0 & 1
\end{pmatrix} = \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}
\]

4.8 Eigenvalues and Eigenvectors

\[
\lambda = \frac{A + B \pm \sqrt{(A - B)^2 + 4H^2}}{2}
\]

(1)

Since argument in the square root is a sum of squares it is positive and hence both eigenvalues are real.

For \( A = 4, B = 1, \) and \( H = 2 \):

\[
\begin{align*}
\lambda_+ &= 5 \\
\lambda_- &= 0
\end{align*}
\]

Take the dot product of the two eigenvectors to show that they are orthogonal.

\[
v_+ \cdot v_- = \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \end{pmatrix} = 1 - 1 = 0
\]

Therefore the eigenvectors are orthogonal.