6 Inverse Functions

6.1 Inverse Functions

Find $(f^{-1})'(8)$.

\[ f(x) = 9 - x^2, \quad 0 \leq x \leq 3 \]

6.2 Exponential Functions and Their Derivatives

Find $dy/dx$.

\[ e^{x/y} = x - y \]

6.3 Logarithmic Functions

Solve the equation for $x$:

\[ \ln x + \ln(x - 1) = 1 \]
6.4 Derivatives of Logarithmic Functions

Find \( G'(y) \).

\[
G(y) = \ln \left( \frac{(2y + 1)^5}{\sqrt{y^2 + 1}} \right)
\]

6.5 Exponential Growth and Decay

A bacteria culture initially contains 100 cells and grows at a rate proportional to its size. After an hour the population has increased to 240. (a) Find an expression for the number of bacteria after \( t \) hours. (b) When will the population reach 10,000? Express your answers without estimating them!

6.6 Inverse Trigonometric Functions

Evaluate the Integral.

\[
\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{8}{1 + x^2} \, dx
\]

6.7 Hyperbolic Functions

Find \( f'(x) \).

\[
f(x) = \sinh(\ln x)
\]

6.8 Indeterminate Forms and L'Hospital’s Rule

Find the limits.

\[
\lim_{x \to (\pi/2)^+} \frac{\cos x}{1 - \sin x}
\]

\[
\lim_{x \to 0^+} \left( \frac{1}{x} - \frac{1}{e^x - 1} \right)
\]

7 Techniques of Integration

7.1 Integration by Parts

Evaluate the integral.

\[
\int (x^2 + 2x) \cos x \, dx
\]

7.2 Trigonometric Integrals

Evaluate the integral.

\[
\int \sin^2 x \cos^3 x \, dx
\]

7.3 Trigonometric Substitution

Evaluate the integral.

\[
\int \frac{\sqrt{1 + x^2}}{x} \, dx
\]
7.4 Integration of Rational Functions by Partial Fractions
Evaluate the integral.
\[ \int \frac{ax}{x^2 - bx} \, dx \]

7.5 Strategy for Integration
Evaluate the integral.
\[ \int \arctan \sqrt{x} \, dx \]

7.6 Integration using Tables and Computer Algebra Systems
Evaluate the integral
\[ \int \frac{\cos x}{\sin^2 x - 9} \, dx \]
using
\[ \int \frac{du}{u^2 - a^2} = \frac{1}{2a} \ln \left| \frac{u - a}{u + a} \right| + C \]

7.7 Approximate Integration
Use the Trapezoid Rule to approximate with \( n = 8 \). Just write down all the terms in the summation.
\[ \int_0^4 x^3 \sin x \, dx \]

7.8 Improper Integrals
Determine whether the integral is convergent or divergent. Evaluate the integral if it is convergent.
\[ \int_3^\infty \frac{1}{(x - 2)^{3/2}} \, dx \]

8 Further Applications of Integration

8.1 Arc Length
Set up an integral that represents the length of the curve.
\( y = x - \ln x, \ 1 \leq x \leq 4 \)

8.2 Area of a Surface of Revolution
Find the exact area of the surface obtained by rotating the curve about the \( x \)-axis.
\( y = x^3, \ 0 \leq x \leq 2 \)

Find the area of the surface obtained by rotating the curve about the \( y \)-axis.
\( y = \frac{1}{3} x^{3/2}, \ 1 \leq x \leq 12 \)
8.3 Applications to Physics and Engineering
Sketch the region bounded by the curves \( y = 2x, \ y = 0, \) and \( x = 1. \) Then find the exact coordinates of the centroid.

10 Parametric Equations and Polar Coordinates

10.1 Curves Defined by Parametric Equations
Sketch the curve by using the parametric equations to plot points. Indicate with an arrow the direction in which the curve is traced as \( t \) increases. Eliminate the parameter to find a Cartesian equation of the curve.

\[
x = t^2 - 3, \ y = t + 2, \ -3 \leq t \leq 3
\]

10.2 Calculus with Parametric Curves
Find an equation of the tangent to the curve at the point corresponding to the given value of the parameter.

\[
x = t^3 + 1, \ y = t^4 + t, \ t = -1
\]

10.3 Polar Coordinates
Identify the curve by finding a Cartesian equation for it.

\[r = 5 \cos \theta\]

10.4 Areas and Lengths in Polar Coordinates
Sketch the curve and find the area that it encloses.

\[r = 2 \sin \theta\]

11 Infinite Sequences and Series

11.1 Sequences
Determine whether the sequence converges or diverges. If it converges, find the limit.

\[a_n = n^2 e^{-n}\]

11.2 Series
Determine whether the series is convergent or divergent.

\[
\sum_{n=1}^{\infty} \frac{2+n}{1-2n}
\]

11.3 The Integral Test and Estimates of Sums
Use the Integral Test to determine whether the series is convergent or divergent.

\[
\sum_{n=2}^{\infty} \frac{1}{n \ln n}
\]
11.4 The Comparison Tests
Determine whether the series is convergent or divergent.
\[ \sum_{n=1}^{\infty} \frac{9^n}{3 + 10^n} \]

11.5 Alternating Series Test
Test the series for convergence or divergence.
\[ \sum_{n=1}^{\infty} (-1)^n e^{-n} \]

11.6 Absolute Convergence and the Ratio and Root Tests
Determine whether the series is absolutely convergent, conditionally convergent, or divergent.
\[ \sum_{n=1}^{\infty} \frac{\sin n}{2^n} \]

11.7 Strategy for Testing Series
Test the series for convergence or divergence.
\[ \sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n}}{(2n)!} \]

Test the series for convergence or divergence.
\[ \sum_{n=1}^{\infty} \frac{n!}{e^{n^2}} \]

11.8 Power Series
Find the radius of convergence of the series.
\[ \sum_{n=1}^{\infty} \frac{x^n}{n^4 4^n} \]

11.9 Representations of Functions as Power Series
Find a power series representation for the function and determine the radius of convergence.
\[ f(x) = \ln(5 - x) \]

11.10 Taylor and Maclaurin Series
Use the Maclaurin series in Table 1 (see below) to obtain the Maclaurin series for the function.
\[ f(x) = x \cos(2x) \]
\[ \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \]
6 Inverse Functions - Solutions

6.1 Inverse Functions

\[(f^{-1})'(8) = \frac{-1}{2}\]

6.2 Exponential Functions and Their Derivatives

\[
\frac{dy}{dx} = \frac{y^2 - ye^{x/y}}{y^2 - xe^{x/y}}
\]

6.3 Logarithmic Functions

\[x = \frac{1 \pm \sqrt{1 + 4e^2}}{2}\]

6.4 Derivatives of Logarithmic Functions

\[G'(y) = \frac{10}{2y+1} - \frac{y}{y^2+1}\]

6.5 Exponential Growth and Decay

1. \[P = 100e^{\ln(2.4)t}\] or \[P = 100 \cdot 2.4^t\]

2. \[t = \frac{\ln(100)}{\ln(2.4)}\]

6.6 Inverse Trigonometric Functions

\[I = \frac{4\pi}{3}\]

6.7 Hyperbolic Functions

\[f'(x) = \cosh(\ln x) \cdot \frac{1}{x}\]

6.8 Indeterminate Forms and L’Hospital’s Rule

The limit DNE (it goes to \(-\infty\)).

\[L = \frac{1}{2}\]
7 Techniques of Integration - Solutions

7.1 Integration by Parts

\[ I = (x^2 + 2x) \sin x + (2x + 2) \cos x - 2 \sin x + C \]

7.2 Trigonometric Integrals

\[ I = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C \]

7.3 Trigonometric Substitution

\[ I = \ln \left( \frac{\sqrt{1 + x^2}}{x} - \frac{1}{x} \right) + \sqrt{x^2 + 1} + C \]

7.4 Integration of Rational Functions by Partial Fractions

We can simplify and take the integral:

\[ I = a \ln(x - b) + C \]

7.5 Strategy for Integration

\[ I = x \arctan \sqrt{x} - \sqrt{x} + \arctan \sqrt{x} + C \]

7.6 Integration using Tables and Computer Algebra Systems

\[ I = \frac{1}{6} \ln \left| \frac{\sin x - 3}{\sin x + 3} \right| + C \]

7.7 Approximate Integration

\[ I \approx \frac{1}{2} \left[ (0)^3 \sin 0 + 2 \left( \frac{1}{2} \right)^3 \sin \frac{1}{2} + 2 (1)^3 \sin 1 + 2 \left( \frac{3}{2} \right)^3 \sin \frac{3}{2} \right. \\
\left. + 2 (2)^3 \sin 2 + 2 \left( \frac{5}{2} \right)^3 \sin \frac{5}{2} + 2 (3)^3 \sin 3 + 2 \left( \frac{7}{2} \right)^3 \sin \frac{7}{2} + (4)^3 \sin 4 \right] \]

7.8 Improper Integrals

It is convergent. \( I = -2 \)

8 Further Applications of Integration - Solutions

8.1 Arc Length

\[ L = \int_{1}^{4} \sqrt{1 + \left( 1 - \frac{1}{x} \right)^2} \, dx \]
8.2 Area of a Surface of Revolution

\[ SA = \frac{16\pi}{3} \]

\[ SA = 64\pi \left( \frac{32}{5} + \frac{8}{3} - \frac{5\sqrt{5}}{32} + \frac{5\sqrt{5}}{24} \right) \]

8.3 Applications to Physics and Engineering

The coordinates of our centroid are \((\frac{2}{3}, \frac{2}{3})\).

10 Parametric Equations and Polar Coordinates - Solutions

10.1 Curves Defined by Parametric Equations

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<tr>
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<td>6</td>
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</tr>
</tbody>
</table>
10.2 Calculus with Parametric Curves

\[ y = \pm \sqrt{x + 3 + 2} \]

10.3 Polar Coordinates

\[ (x - \frac{5}{2})^2 + y^2 = \left( \frac{5}{2} \right)^2 \]

This is a circle with center \( \left( \frac{5}{2}, 0 \right) \) and radius \( \frac{5}{2} \).

10.4 Areas and Lengths in Polar Coordinates

\[ A = \pi \]

11 Infinite Sequences and Series - Solutions

11.1 Sequences

\[ L = 0 \]

This limit exists, so this sequence converges.
11.2 Series
The series is divergent.

11.3 The Integral Test and Estimates of Sums
The series is divergent.

11.4 The Comparison Tests
The series is convergent.

11.5 Alternating Series Test
The series is convergent.

11.6 Absolute Convergence and the Ratio and Root Tests
The series is absolutely convergent.

11.7 Strategy for Testing Series
The series is absolutely convergent.

11.8 Power Series
The radius of convergence is 4.

11.9 Representations of Functions as Power Series

\[
\ln(5 - x) = - \sum_{n=1}^{\infty} \frac{x^n}{n \cdot 5^n}
\]

The radius of convergence is 5.

11.10 Taylor and Maclaurin Series

\[
f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{4^n x^{2n+1}}{(2n)!}
\]