M119 Exam Jam

Spring 2017

Solutions Packet

1. Find the average rate of change of $f(x) = x^2 - 3$ between $x = 1$ and $x = 5$

   The average rate of change is equal to change in $f(x)$ divided by change in x.

$$\frac{f(5) - f(1)}{5 - 1} = \frac{(25-3) - (1-3)}{4} = \frac{22+2}{4} = \boxed{6}$$

2. Find the instantaneous rate of change of $f(x) = x^4 - 3x^2 + 2$ at $x = 4$

   The instantaneous rate of change is found by computing the derivative of the given function and evaluating the derivative at the given value of x.

$$f'(x) = 4x^3 - 6x|_{x=4} \longrightarrow f'(4) = 4(64) - 6(4) = 256 - 24 = \boxed{232}$$

3. Find $f'(-1)$ if $f(x) = x^5 - 3x^4 + 1$

   Find the derivative of the function and evaluate at $x = -1$.

$$f'(x) = 5x^4 - 12x^3|_{x=-1} \longrightarrow f'(-1) = 5 + 12 = \boxed{17}$$

4. Differentiate: $y = 8\sqrt{x} - \frac{2}{x^3}$

   First, write each term in exponent form.

$$y = 8x^{1/2} - 2x^{-3}$$

   Then, evaluate using the Power Rule

$$\frac{dy}{dx} = 4x^{-1/2} + 6x^{-4} \longrightarrow \boxed{\frac{dy}{dx} = \frac{4}{\sqrt{x}} + \frac{6}{x^4}}$$

5. If $g(t) = e^{-4t}$ find $g"(0)$

   Differentiate the function twice and then evaluate at $x = 0$.

$$g(t) = e^{-4t} \longrightarrow g'(t) = -4e^{-4t} \longrightarrow g"(t) = 16e^{-4t}|_{x=0} \longrightarrow 16e^{-4(0)} = \boxed{16}$$

6. Find $\frac{d^2y}{dx^2}|_{x=4}$ if $y = lnx$

From the formula, we know that:

$$\frac{d}{dx}(lnx) = \frac{1}{x} = x^{-1}$$

Differentiate again and evaluate at $x = 4$.

$$\frac{d^2y}{dx^2} = -x^{-2}|_{x=4} = \boxed{-\frac{1}{16}}$$

7. Find the slope of the line tangent to $f(x) = x^4 + 7x - 8$ at $x = 0$

The slope of the tangent line to $f(x)$ at $x = 0$ is equal to $f'(x)|_{x=0}$

$$f'(x) = 4x^3 + 7 \longrightarrow f'(0) = 4(0) + 7 = \boxed{7}$$

8. Find an equation for the line tangent to the graph of $y = 3x^2 - x$ at $x = -1$

First, find the slope of the tangent line to y at $x = -1$ by finding the derivative and evaluating f'(-1).

$$\frac{dy}{dx} = 6x - 1|_{x=-1} \longrightarrow 6(-1) - 1 = \boxed{-7}$$

Next, find a point on y corresponding to $x = -1$.

$$y(-1) = 3 + 1 = \boxed{4}$$

Last, use point-slope form to write the equation of the line

$$y_2 - y_1 = m(x_2 - x_1) \longrightarrow y - 4 = -7(x + 1)$$

$$\boxed{y = -7x - 3}$$

9. Find an equation for the line tangent to the graph of $f(x) = e^{3x} + 2x + 1$ at $x = 0$

Find the slope by taking the derivative and evaluating f'(0)

$$f'(x) = 3e^{3x} + 2 \longrightarrow f'(0) = 3 + 2 = \boxed{5}$$

Next, find a point on $f(x)$ corresponding to $x = 0$

$$f(0) = 1 + 2(0) + 1 = \boxed{2}$$

Use point-slope form to write the equation

$$y - 2 = 5(x - 0) \longrightarrow \boxed{y = 5x + 2}$$

10. Given that $y = 5(x^4 - 2)^3$, find $\frac{dy}{dx}\,|_{x=1}$

First, use the Chain Rule to differentiate.

$$\frac{dy}{dx} = 3 \cdot 5(x^4 - 2)^2 \cdot 4x^3 = 60x^3(x^4 - 2)^2$$

Then, evaluate $\frac{dy}{dx}$ at $x = 1$

$$y'(1) = 60(1-2)^2 = 60(1) = \boxed{60}$$

11. Given $y = 2\ln(5x^3 - x)$, find $\frac{dy}{dx}$

Use the Chain Rule to differentiate.

$$\frac{dy}{dx} = \frac{2}{5x^3 - x} \cdot (15x^2 - 1) = \boxed{\frac{30x^2 - 2}{5x^3 - x}}$$

12. If $P(t) = 200te^{0.04t}$, find $P'(t)$

Use the Product Rule and the Chain Rule to differentiate

$$P'(t) = 200e^{0.04t} + (0.04)(200)te^{0.04t} = \boxed{200e^{0.04t} + 8te^{0.04t}}$$

13. Given $f(z) = z^5 \ln z$, find $f'(z)$

Use the Product Rule to find the derivative

$$f'(z) = 5z^4 \cdot \ln z + z^5 \cdot \frac{1}{z} = \boxed{5z^4 \ln z + z^4}$$

14. Find all points where the tangent line is horizontal: $f(x) = x^3 - 3x^2 - 9x$

The tangent line is horizontal when its slope is 0. Since the derivative is equal to the slope of the tangent line, find the derivative and set it equal to 0. Plug the corresponding x-values back into $f(x)$ to find the points.

$$f'(x) = 3x^2 - 6x - 9 = 0 \longrightarrow 3(x^2 - 2x - 3) = 0 \longrightarrow 3(x-3)(x+1) = 0 \longrightarrow \boxed{x = 3, -1}$$

$$f(3) = 27 - 3(9) - 9(3) = -27$$

$$f(-1) = -1 - 3 + 9 = 5$$

The points are: $\boxed{(3, -27)}$ and $\boxed{(-1, 5)}$

15. Given $f(x) = 16x - x^2$, find all points where the tangent line is horizontal.

$$f'(x) = 16 - 2x = 0 \longrightarrow -2x = -16 \longrightarrow x = 8$$

$$f(8) = 16(8) - 64 = 64$$

The point at which the tangent line is horizontal: $(8, 64)$

16. For $f(x) = x^4 + 4x^3 + 10$, find the critical points, and then determine if each point is a local minimum, local maximum, or neither.

The critical points are the points at which $f'(x) = 0$.

$$f'(x) = 4x^3 + 12x^2 = 0 \longrightarrow 4x^2(x + 3) = 0 \longrightarrow x = 0, -3$$

This yields the following 2 critical points: $\boxed{(0, 10)}$ and $\boxed{(-3, -17)}$.

In order to determine local maximum or minimum, use a number line and evaluate the sign derivative on either side of each critical point.

Based on the number line: $(-3, -17)$ is a local minimum and $(0, 10)$ is neither.

17. Given that $g(t) = t^3 - 3t^2 + 3t - 2$, find the inflection points.

First, differentiate the function twice to find the 2nd derivative.

$$g'(t) = 3t^2 - 6t + 3$$
$$g''(t) = 6t - 6$$

Set the 2nd derivative equal to 0 and solve for associated t-values

$$g''(t) = 6t - 6 = 0 \longrightarrow \boxed{t = 1}$$

Last, plug $t = 1$ into the original function to find the point.

$$g(1) = 1^3 - 3(1)^2 + 3(1) - 2 = -1 \longrightarrow \boxed{(1, -1)}$$

18. Find the absolute maximum and absolute minimum values of the function on the given interval: $f(x) = x^2 - 10x$ on the interval $[0, 6]$

First, evaluate the function at each end of the given interval.

$$f(0) = \boxed{0}$$
$$f(6) = 36 - 60 = \boxed{-24}$$

Next, find the critical points of the function

$$f'(x) = 2x - 10 = 0$$
$$2x = 10 \longrightarrow x = 5$$
$$f(5) = 25 - 50 = \boxed{-25}$$

Based on these values, $(0, 0)$ is an absolute maximum and $(5, -25)$ is an absolute minimum. Additionally, we know that these are absolute minima (as opposed to local minima) because we are evaluating the function within an interval.

19. At a price of \$20 per ticket, a group can fill every seat in a theater with 930 seats. For every additional dollar charged, the number of people buying tickets decreases by 30.

a. Find the revenue function (as a function of price).

First, we need to find a function that relates ticket price to the number of people buying the tickets. Let p represent ticket price and x represent the number of people buying tickets.

We know that every 1 dollar increase in price leads to a decrease in attendance by 30. This means that the function is linear and its slope is $m = -30$.

We also know that a 20 dollar ticket price leads to a maximum attendance of 930. This information gives the point $(20, 930)$. Now use point-slope form to find the equation of the line.

$$x - 930 = -30(p - 20)$$
$$x = -30p + 1530$$

Note: x is the dependent variable in this equation.

$$R(p) = x \cdot p \longrightarrow \boxed{R(p) = -30p^2 + 1530p}$$

b. Find the ticket price that maximizes the revenue.

To find the ticket price that maximizes revenue, find the derivative and set it equal to 0.

$$R'(p) = -60p + 1530 = 0$$
$$60p = 1530$$
$$\boxed{p = 25.5}$$

c. What is the maximum revenue?

Now plug in the answer to part b to find the maximum revenue.

$$R(25.5) = \boxed{19,507.50}$$

20. A company finds that the demand equation for a quantity q of Jphones sold at price p, in dollars is $p = 870 - 3q$. To produce these Jphones, the company finds that fixed costs are \$2875 and the variable cost per unit is \$126.

a. At what quantity is the profit maximized?

First, find the revenue function using the given demand equation.

$$R(q) = p \cdot q \longrightarrow R(q) = 870q - 3q^2$$

Next, find the cost function using the given information.

$$C = 2875 + 126q$$

Now, find the profit function.

$$P = R - C \longrightarrow P(q) = -3q^2 + 744q - 2875$$

Last, maximize the profit function by differentiating and setting the derivative to 0.

$$P'(q) = -6q + 744 = 0 \longrightarrow \boxed{q = 124}$$

b. What is the maximum profit?

Plug $q = 124$ into the profit equation to find the maximum profit.

$$P(124) = \boxed{42,253}$$

21. If the population of a town doubled in 15 years, find the continuous annual growth rate. Write your answer as a percent.

If $T_d = 15$ years:

$$P = P_0 e^{T_d k}$$
$$2 = 1e^{15k}$$
$$\ln 2 = 15k$$
$$k = 0.0462 = \boxed{4.62\%}$$

22. If money is invested in an account that pays interest compounded continuously at 2.9% per year, how long will it take for the investment to double. Write your answer with 1 decimal place and include units.

$$P = P_0 e^{T_d k}$$
$$2 = 1e^{0.029T_d}$$
$$\ln 2 = 0.029T_d$$
$$\boxed{T_d = 23.9 \text{ years}}$$

23. If the half-life of a medication is 9 hours, find the rate of decay.

$$k = \frac{-\ln 2}{H}$$
$$= \frac{-\ln 2}{9}$$
$$= -0.077 = \boxed{-7.7\%}$$

note: the value for k is negative because it represents decay.

24. If the decay rate for a substance is 4.2% per week, find the half-life. Give your answer with 1 decimal place and include units.

$$H = \frac{-\ln 2}{k}$$
$$= \frac{-\ln 2}{-0.042}$$
$$\boxed{H = 16.5 \text{ weeks}}$$

25. Find each of the following indefinite integrals:

a. $\int (e^{5t} + t^5)dt$

$$= \boxed{\frac{1}{5}e^{5t} + \frac{1}{6}t^6 + C}$$

b. $\int \frac{2}{x^5} dx$

$$= \int 2x^{-5}dx = \boxed{-\frac{1}{2}x^{-4} + C}$$

c. $\int (\frac{1}{x^4} - \frac{4}{x})dx$

$$= \int x^{-4} - 4x^{-1} = \boxed{-\frac{1}{3}x^{-3} - 4\ln x + C}$$

26. Evaluate each of the following definite integrals:

    a. $\int_1^2 2t^4 dt$

$$= \frac{2}{5}t^5|_1^2 = \frac{2}{5}(32-1) = \boxed{\frac{62}{5}}$$

    b. $\int_{-1}^1 (4x^3 - 1)dx$

$$= x^4 - x\Big|_{-1}^1 = \boxed{-2}$$

    c. $\int_{16}^{36} 3\sqrt{x}dx$

$$2x^{\frac{3}{2}}\Big|_{16}^{36} = 2(6^3 - 4^3) = \boxed{304}$$

    d. $\int_1^e \frac{5}{x}dx$

$$5 \cdot \ln x\Big|_1^e = 5 - 5\ln 1 = \boxed{5}$$

27. The marginal cost function of a product, in dollars per unit, is $C'(x) = 6x^2 - 60x + 10$. Find the total cost function if fixed costs are \$4,000.

$$C(x) = \int C'(x)dx + C_0$$
$$C(x) = \int (6x^2 - 60x + 10)dx + 4000$$
$$C(x) = \boxed{2x^3 - 30x^2 + 10x + 4000}$$

28. Find the area of the region bounded by $y = x^3 + 3$ and the x-axis over the interval $[0, 2]$

$$A = \int_0^2 (x^3 + 3)dx$$
$$A = \frac{1}{4}x^4 + 3x\Big|_0^2 = \boxed{10}$$

29. Find the area of the region bounded by $f(x) = 9 - x^2$ over the interval $[-3, 3]$

$$A = \int_{-3}^3 (9 - x^2)dx$$
$$A = \left(9x - \frac{x^3}{3}\right)\Bigg|_{-3}^3$$
$$A = (27 - \frac{27}{3}) - (-27 + \frac{27}{3}) = \boxed{36}$$

30. Your business estimates that sales are growing continuously at a rate given by $S'(t) = 3t^2 + 2$, where $S'(t)$ is given in dollars per day on day t. Find the accumulated sales for the first 5 days.

$$\begin{aligned} S(t) &= \int_0^5 S'(t)dt \\ &= \int_0^5 (3t^2 + 2)dt \\ &= (t^3 + 2t)\Big|_0^5 \\ &= (5)^3 + 2(5) - 0 = \boxed{\$135} \end{aligned}$$

31. Find the present value of \$7,000 due 8 years from now if interest is compounded continuously at a rate of 2.5% per year.

$$PV = \frac{A}{e^{rt}}$$

$$PV = \frac{7000}{e^{(0.025)(8)}}$$

$$PV = \boxed{\$5,737.71}$$

32. Find the present value of a continuous income stream of \$7,000 per year for 8 years if interest is compounded continuously at a rate of 2.5% per year.

$$A = 7000 \cdot \int_0^8 (e^{(-0.025)t})dt$$

Let $u = -0.025t$ and $du = -0.025dt$

$$A = -\frac{7000}{0.025} \cdot e^u du$$

$$A = -280,000 \cdot e^{(-0.025)t}\Big|_0^8$$

$$A = \boxed{\$50,755.39}$$