M118 Exam Jam Solutions

Contents

Chapter 2: Set Theory

1. Let $U = \{a, b, c, d, e, f\}$, $A = \{a, e\}$, $B = \{b, c, d, e\}$, and $C = \{c, f\}$. Find the following.
   (a) $(A \cap B) \cup C$
   (b) $A \cap (B \cup C)$
   (c) $A \times B'$

Solution

(a) $(A \cap B) \cup C$

First, find $(A \cap B)$. $(A \cap B) = \{e\}$. Then combine the elements in both sets to find $(A \cap B) \cup C = \boxed{\{c,\ e,\ f\}}$.

(b) $A \cap (B \cup C)$

First, find $(B \cup C)$. $(B \cup C) = \{b,\ c,\ d,\ e,\ f\}$. Then find the elements that the sets have in common to find $A \cap (B \cup C) = \boxed{\{e\}}$.

(c) $A \times B'$

First, find B'. $B' = \{a,\ f\}$. Then pair the elements in A with the elements in B' to create a set of ordered pairs, where $A \times B' = \boxed{\{(a, a), (a, f), (e, a), (e, f)\}}$.

2. Of the 72 students in a mathematics class, 37 are taking French, 32 are taking German, and 9 are taking both French and German. How many students are taking neither French nor German?

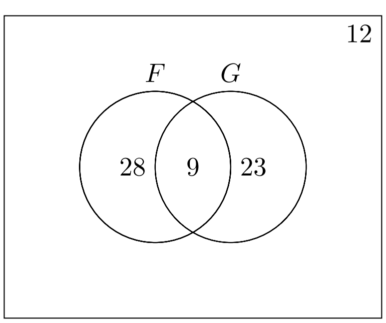
Solution

Let $n(F)$ = the number of students taking French and $n(G)$ = the number of students taking German. We can find the number of students that are taking only French $(F - G)$ and only German $(G - F)$ by doing the following calculations.

$$n(F) - n(F \cap G) = n(F - G)$$
$$37 - 9 = 28$$

$$n(G) - n(F \cap G) = n(G - F)$$
$$32 - 9 = 23$$

Now that we know how many students are in each class, we can create a venn diagram to represent the situation. We can then use $n(F \cup G)$ to find how many students are taking neither French nor German.



$$n(F \cup G) = 28 + 9 + 23 = 60$$
$$n(U) - n(F \cup G) = n(F \cup G)'$$
$$72 - 60 = 12$$

$\boxed{\text{12 students are taking neither French nor German.}}$

3. A sample of 100 students was surveyed about their preferences for types of music played on the radio. The choices to select from were rock, blues, and country. The results of the survey are listed below:

72 like rock	22 like rock and blues	27 like country and blues
40 like blues	20 like rock and country	12 like all three
37 like country		

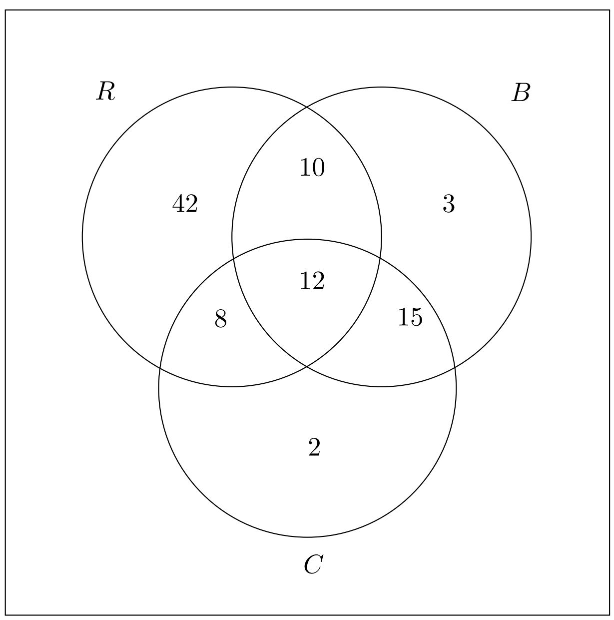
(a) How many students surveyed like only one of the three types of music?

(b) How many students surveyed like none of the three types of music?

(c) How many students surveyed like at least two of the three types of music?

Solution

Let $n(R)$ represent the number of students that prefer rock, $n(B)$ represent the number of students that prefer blues and $n(C)$ represent the number of students that prefer country. Create a venn diagram of the situation.



(a) How many students surveyed like only one of the three types of music?
$42 + 3 + 2 = \boxed{47}$

(b) How many students surveyed like none of the three types of music?
$n(U) - n(R \cup B \cup C) = n(R \cup B \cup C)'$
$100 - 92 = \boxed{8}$

(c) How many students surveyed like at least two of the three types of music?
$10 + 8 + 15 + 12 = \boxed{45}$

4. Let A and B be subsets of a universal set U, where:

$$\begin{aligned} n(U) &= 40 \\ n(A' \cap B') &= 8 \\ n(B - A) &= 10 \\ n(A - B) &= 10 \end{aligned}$$

Find $n(A \cap B)$

Solution

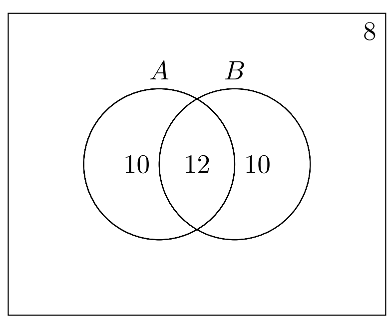
According to DeMorgan's Laws, $n(A' \cap B')$ is equal to $n(A \cup B)'$, which is the complement of the union. Using the information provided in the problem, we can find the union of A and B.

$$\begin{aligned} n(U) &= n(A \cup B) + n(A \cup B)' \\ n(U) &= n(A \cup B) + n(A' \cap B') \\ 40 &= n(A \cup B) + 8 \\ 32 &= n(A \cup B) \end{aligned}$$

After finding $n(A \cup B)$, we can find the intersection by subtracting the values in "only A" and "only B" from the union.

$$\begin{aligned} n(A \cup B) &= n(A - B) + n(B - A) + n(A \cap B) \\ n(A \cup B) - n(A - B) - n(B - A) &= n(A \cap B) \\ 32 - 10 - 10 &= n(A \cap B) \end{aligned}$$

$$\boxed{12 = n(A \cap B)}$$



Chapter 3: Combinatorics

5. How many license plates can be produced using five symbols on each plate where the first two symbols are different letters of the alphabet and the following three symbols are different numbers selected from the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$? No number or letter can be repeated on the same license plate.

Solution

We can use the fundamental counting principle to solve this problem because the order matters. Since there are 26 letters in the alphabet, we have 26 choices for the first letter and 25 choices for the second letter because no letters can be repeated. We also have 10 choices for the first number, then 9, and 8 for the third number, because no numbers can be repeated either.

$$26 \cdot 25 \cdot 10 \cdot 9 \cdot 8 = \boxed{468,000 \text{ license plates}}$$

6. How many distinguishable arrangements can be made from all the letters in the word MATHEMATICS?

Solution

Within the word MATHEMATICS, we can see that there are 11 letters total and that the letters, A, M, and T appear twice. To find the number of distinguishable arrangements that can be made, we can use the formula for an ordered permutation. In this formula, n is the total number of letters and k_1 is the number of times the first letter repeats itself, k_2 is the number of times the second letter repeats itself, and so on. If a letter is only listed once, it will not affect the denominator of the Ordered Permutation Formula because it would only multiply it by 1.

Ordered Permutation Formula: $$\frac{n!}{(k_1! \cdot k_2! \cdot k_3! \cdot ... \cdot k_m!)}$$

$$\frac{11!}{(2! \cdot 2! \cdot 2!)}$$

$$\frac{39,916,800}{8}$$

$$\boxed{4,989,600}$$

7. How many committees of four can be formed from 20 Republicans and 15 Democrats if

   (a) two members of each party must be on the committee?

   (b) at least one Republican and at least one Democrat must be on the committee?

Solution

(a) two members of each party must be on the committee?

We can solve this using combinations because the order in which the members are selected does not matter.

$$\mathrm{C}(20,2)\cdot\mathrm{C}(15,2) = (190)(105) = \boxed{19,950}$$

(b) at least one Republican and at least one Democrat must be on the committee?

Since we are finding the number of ways of choosing at least one Republican and at least one Democrat, we have to add three groups of combinations of Republicans and Democrats where there are a total of four members. The possible combinations are 3 Republicans and 1 Democrat, 2 Republicans and 2 Democrats, and 1 Republican and 3 Democrats.

$$\begin{aligned}\mathrm{C}(20,3)\cdot\mathrm{C}(15,1)+\mathrm{C}(20,2)\cdot\mathrm{C}(15,2)+\mathrm{C}(20,1)\cdot\mathrm{C}(15,3)\\ (1140)(15)+(190)(105)+(20)(455)\\ \boxed{46,150}\end{aligned}$$

If we were to try to solve this problem using the negative approach, we would need to subtract the number of ways to get all Republicans or all Democrats from the total number of ways.

$$\begin{aligned}\mathrm{C}(35,4)-\mathrm{C}(20,4)-\mathrm{C}(15,4)\\ 52,360-4,845-1,365\\ \boxed{46,150}\end{aligned}$$

8. How many ways can 6 different contracts be distributed amongst 10 different firms, if any one firm can be awarded multiple contracts?

Solution

We can solve this problem by using the Fundamental Counting Principle. The Fundamental Counting Principle is used when there is an experiment consisting of multiple stages (k) that can be performed in n number of ways. In this problem, the different contracts are the stages and the number of ways that they can be distributed is 10 each time, because any one firm can be awarded multiple contracts.

$$\begin{aligned}\text{Fundamental Counting Principle:}\quad & n_1\cdot n_2\cdot \ldots \cdot n_k\\ & 10\cdot 10\cdot 10\cdot 10\cdot 10\cdot 10\\ & 10^6\end{aligned}$$

$$\boxed{1,000,000}$$

9. How many different 3-digit EVEN numbers can be formed from the set$\{1,2,3,4,5,6,7,8\}$ if it cannot start with a 1 or 2 and repetition is allowed?

Solution

The best way to approach this problem is by using the Fundamental Counting Principle. Using that approach, we have to find out how many choices we have for each digit. Since we know that the first digit cannot be a 1 or 2, we are left with 6 choices: 3, 4, 5, 6, 7, or 8.

For the middle digit, we have all 8 choices because there are no restrictions on the middle digit and repetition between digits is allowed.

For the last digit, we know that it must be an even number. This leaves us with 4 choices: 2, 4, 6, or 8. After finding the number of choices per digit, multiply to solve.

$$6 \cdot 8 \cdot 4 = \boxed{192}$$

Chapter 4: Probability

10. Let A and B be disjoint events with $\Pr[A] = 0.55$ and $\Pr[B] = 0.25$. Find $\Pr[A \cup B]$.

Solution

Since A and B are disjoint events they have no intersection. Therefore, their individual probabilities add together to form their union.

$$\Pr[A \cup B] = \Pr[A] + \Pr[B]$$
$$\Pr[A \cup B] = 0.55 + 0.25$$
$$\Pr[A \cup B] = \boxed{0.8}$$

11. Let C and D be events such that $\Pr[C] = 0.8$, $\Pr[D] = 0.4$, and $\Pr[C \cup D] = 0.9$.
    (a) Evaluate $\Pr[D|C]$.
    (b) Evaluate $\Pr[C|D]$.

Solution

(a) Evaluate $\Pr[D|C]$.

To find $\Pr[D|C]$ and $\Pr[C|D]$, we first need to find $\Pr[C \cap D]$. We can find this by using the union formula to solve for $\Pr[C \cap D]$.

$$\text{Union Formula: } \Pr[A \cup B] = \Pr[A] + \Pr[B] - \Pr[A \cap B]$$

$$\begin{aligned}
\Pr[C \cup D] &= \Pr[C] + \Pr[D] - \Pr[C \cap D] \\
0.9 &= 0.8 + 0.4 - \Pr[C \cap D] \\
0.9 &= 1.2 - \Pr[C \cap D] \\
-0.3 &= -\Pr[C \cap D] \\
0.3 &= \Pr[C \cap D]
\end{aligned}$$

Now that we know $\Pr(C \cap D)$, we can use the conditional probability formula to solve for $\Pr[D|C]$.

$$\text{Conditional Probability Formula: } \Pr[A|B] = \frac{\Pr[A \cap B]}{\Pr[B]}, \text{where } \Pr[B] > 0$$

$$\begin{aligned}
\Pr[D|C] &= \frac{\Pr[D \cap C]}{\Pr[C]} \\
&= \frac{0.3}{0.8} \\
&= \boxed{\frac{3}{8}}
\end{aligned}$$

(b) Evaluate $\Pr[C|D]$.

Using the calculations in part (a), we can solve for $\Pr[C|D]$ using the conditional probability formula.

$$\begin{aligned}
\Pr[C|D] &= \frac{\Pr[C \cap D]}{\Pr[D]} \\
&= \frac{0.3}{0.4} \\
&= \boxed{\frac{3}{4}}
\end{aligned}$$

12. Five cards are drawn at random from a standard deck of 52. What is the probability that all five cards are of the same suit?

Solution

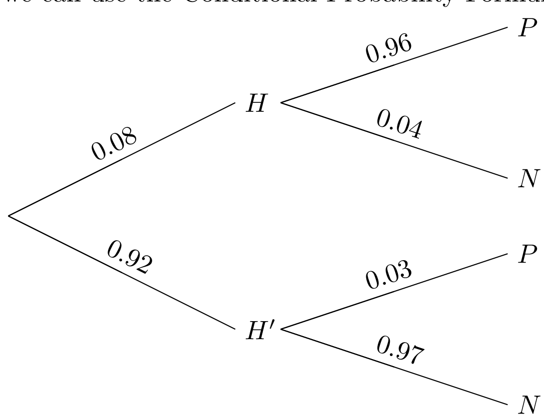
We can solve this problem using combinations because the order in which we choose the cards does not matter. To find the probability that all five cards are of the same suit, we first need a combination that determines the suit. This is multiplied by a combination that determines the rank of the cards selected. Since there are four suits and thirteen ranks, our combinations are C(4, 1) and C(13, 5), because we are choosing five cards. To determine the probability, we also need to divide these combinations by the total probability which is C(52, 5) because there are 52 cards in the deck total.

$$\frac{\mathrm{C}(4,1)\cdot\mathrm{C}(13,5)}{\mathrm{C}(52,5)} = \frac{5148}{2,598,960} \approx \boxed{0.00198}$$

13. It is believed that 8% of the population has hepatitis. A medical firm has a new test to detect hepatitis. It was found that if a person has hepatitis, the test will detect it (show a positive result) in 96% of the cases; it was also found that it will show a positive result in 3% of those who do not have hepatitis. What is the probability that a person has hepatitis given that they tested positive?

Solution

The best way to approach this problem is by first creating a tree diagram. After creating the tree diagram, we can use the Conditional Probability Formula to solve.



Conditional Probability Formula: $\Pr[A|B] = \dfrac{\Pr[A \cap B]}{\Pr[B]}$, where $\Pr[B] > 0$

$$\Pr[H|P] = \frac{\Pr[H \cap P]}{\Pr[P]}$$

$$\Pr[H|P] = \frac{(0.08)(0.96)}{(0.08)(0.96) + (0.92)(0.03)} = \frac{0.0768}{0.1044} \approx \boxed{0.7356}$$

14. An unfair coin with $\Pr[H] = 0.25$ is flipped. If the flip results in a head, a student is selected at random from a class of ten boys and twelve girls. Otherwise, a student from a different class containing six boys and five girls is selected.

   (a) What is the probability of selecting a girl?

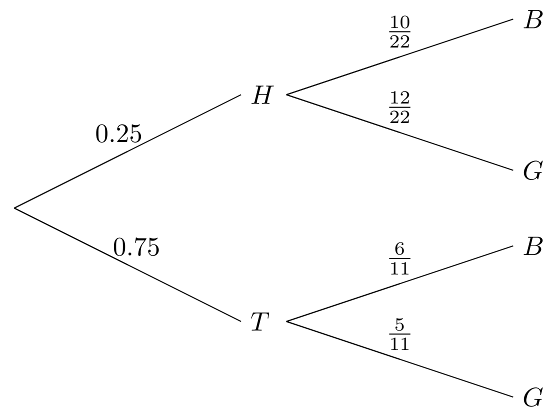
   (b) What is the probability of selecting a girl given that the flip resulted in heads?

   (c) What is the probability of flipping heads given that a girl was selected?

Solution

(a) What is the probability of selecting a girl?

To begin this problem, we first need to construct a tree diagram of the situation. Note that you can find the probability for the second branches by taking the number of boys or girls and putting it over the total number of students in the class.



To find the probability of selecting a girl, we have to add the probability of the intersection of two events: the probability that girls and heads were selected and the probability that girls and tails were selected. By converting the probabilities of heads and tails to fractions, we can find the exact probabilities.

$$\Pr[G] = \Pr[G \cap H] + \Pr[G \cap T]$$

$$= \left(\frac{1}{4}\right)\left(\frac{12}{22}\right) + \left(\frac{3}{4}\right)\left(\frac{5}{11}\right)$$

$$= \frac{12}{88} + \frac{15}{44}$$

$$= \frac{21}{44}$$

$$\approx \boxed{0.4773}$$

(b) What is the probability of selecting a girl given that the flip resulted in heads?

To find the probability of selecting a girl given that the flip resulted in heads, we will refer to the tree diagram from part (a), but we will also need to use the conditional probability formula.

$$\text{Conditional Probability Formula: } \Pr[A|B] = \frac{\Pr[A \cap B]}{\Pr[B]}, \text{where } \Pr[B] > 0$$

$$\Pr[G|H] = \frac{\Pr[G \cap H]}{\Pr[H]}$$

$$= \frac{(\frac{1}{4})(\frac{12}{22})}{(\frac{1}{4})}$$

$$= \frac{6}{11}$$

$$\approx \boxed{0.5455}$$

(c) What is the probability of flipping heads given that a girl was selected?

We can use the conditional probability formula for this problem as well.

$$\Pr[H|G] = \frac{\Pr[H \cap G]}{\Pr[G]}$$

$$= \frac{(\frac{1}{4})(\frac{12}{22})}{(\frac{1}{4})(\frac{12}{22}) + (\frac{3}{4})(\frac{5}{11})}$$

$$= \frac{\frac{12}{88}}{\frac{12}{88} + \frac{15}{44}}$$

$$= \frac{\frac{12}{88}}{\frac{42}{88}}$$

$$= \frac{12}{42}$$

$$= \frac{2}{7}$$

$$\approx \boxed{0.2857}$$

15. A basketball player makes free throws with a 0.8 probability. What is the probability that the player will make at least five of the next six free throws?

Solution

We can solve this problem using the Binomial Probability Formula. Since we are finding the probability of at least 5 free throws, we need to add the probabilities that he makes either 5 or 6 free throws.

Binomial Probability Formula: $\mathrm{C}(n,k)\cdot p^k\cdot(1-p)^{n-k}$

$$\begin{aligned}\mathrm{C}(6,5)\cdot 0.8^5\cdot(1-0.8)^{6-5}+\mathrm{C}(6,6)\cdot 0.8^6\cdot(1-0.8)^{6-6}\\ 0.393216+0.262144\\ 0.65536\end{aligned}$$

$\boxed{\text{The probability that the player will make at least five of the next six free throws is approximately 65.5\%.}}$

16. Given that the probability of a male child is 0.52, what is the probability that a family with four children will have at least one male child?

Solution

We can use the Bernoulli Trial method to solve this problem. Since we are finding the probability of having at least one male child, we have to calculate the probability of having one, two, three, or four male children. Instead of adding the the separate probabilities together, a simpler way to find this probability is to use the negative approach, or the complement. The negative approach works by taking the total probability of the genders of the four children, which is 1, and subtracting the probability that there are no male children.

Binomial Probability Formula: $\mathrm{C}(n,k)\cdot p^k\cdot(1-p)^{n-k}$

$$\begin{aligned}1-\mathrm{C}(4,0)\cdot 0.52^0\cdot(1-0.52)^{4-0}\\ 1-0.05308416\\ 0.94691584\end{aligned}$$

$\boxed{\text{The probability that the family will have at least one male child is approximately 94.7\%.}}$

17. A box contains five red and two blue marbles. If two marbles are drawn from the box without replacement, what is the probability that both marbles are the same color?

Solution

To solve this problem, add the number of ways to get all red marbles to the number of ways to get all blue marbles and divide by the total number of possibilities.

$$\frac{C(5,2)+C(2,2)}{C(7,2)}$$

$$\frac{10+1}{21} = \boxed{\frac{11}{21}}$$

Chapter 5: Statistics

18. Two coins are randomly selected from a pocket containing two quarters, one dime, and one nickel. A random variable X is defined as the value of coins (in cents). Find the expected value.

Solution

Use a probability density function to find the expected value.

	Value of X (x)	Probability (p)	$x \cdot p$
QQ	50	$\frac{C(2,2)}{C(4,2)} = \frac{1}{6}$	$\frac{50}{6}$
QD	35	$\frac{C(2,1)C(1,1)}{C(4,2)} = \frac{2}{6}$	$\frac{70}{6}$
QN	30	$\frac{C(2,1)C(1,1)}{C(4,2)} = \frac{2}{6}$	$\frac{60}{6}$
DN	15	$\frac{C(1,1)C(1,1)}{C(4,2)} = \frac{1}{6}$	$\frac{15}{6}$

$$\mu = \frac{50}{6} + \frac{70}{6} + \frac{60}{6} + \frac{15}{6} = \frac{195}{6} = 32.5$$

$$\boxed{\text{The expected value is 32.5 cents.}}$$

19. A teacher says that the top 4% of the class received an A on the last test. The scores were normally distributed with mean 67 and standard deviation 7. Find the minimum score required to get an A.

Solution

This is a backwards z score problem. That means that we know the resulting probability, but we don't know what a is in the z score equation. Typically, at the end of a z score problem where the probability that X is greater than or equal to a, we subtract the z score value from 1. Since this problem has to be worked backwards, we need to start by subtracting 0.04 from 1.

$$1 - 0.04 = 0.96$$

This gives us a value of 0.96. The closest value within the z score chart to 0.96 is 0.9599, which comes from a value of 1.75. We can plug 1.75 into our z-score equation to find a, which is the minimum score required to get an A.

$$\Pr[X \geq a] = \Pr\left[Z \geq \frac{a - \mu}{\sigma}\right]$$

$$\Pr[X \geq a] = \Pr\left[1.75 \geq \frac{a - 67}{7}\right]$$

$$1.75 = \frac{a - 67}{7}$$

$$7 \cdot (1.75) = 7 \cdot \left(\frac{a - 67}{7}\right)$$

$$12.25 = a - 67$$

$$\boxed{a = 79.25}$$

20. The mean length of angelfish (a popular tropical fish for aquariums) is 10.2 cm with a standard deviation of 2.1 cm. Assuming that the length of angelfish is a normal random variable, find the percent of tropical fish between 7 and 13 cm in length.

Solution

Given a mean of 10.2 and a standard deviation of 2.1, convert the random variables to Z-scores. Then use Appendix B to find the probabilities.

$$\begin{aligned}
\Pr[a \le X \le b] &= \Pr\left[\frac{a-\mu}{\sigma} \le Z \le \frac{b-\mu}{\sigma}\right] \\
\Pr[7 \le X \le 13] &= \Pr\left[\frac{7-10.2}{2.1} \le Z \le \frac{13-10.2}{2.1}\right] \\
&= \Pr[-1.52 \le Z \le 1.33] \\
0.9082 - 0.0643 &= \boxed{0.8439}
\end{aligned}$$

21. Suppose 10% of all people are left-handed. If 300 people are selected at random, find the expected number of left-handed people in the sample. What is the standard deviation?

Solution

Since this problem only involves two possibilites (left-handed or right-handed) it is considered a binomial. We can use the binomial formulas to find the expected value and the standard deviation, where n is the number of people and p is the probability that they are left-handed.

$$\begin{aligned}
E[X] &= n \cdot p \\
E[X] &= 300 \cdot 0.10
\end{aligned}$$

$$\boxed{E[X] = 30}$$

$$\begin{aligned}
\sigma &= \sqrt{n \cdot p(1-p)} \\
\sigma &= \sqrt{300 \cdot 0.10(1-0.10)}
\end{aligned}$$

$$\boxed{\sigma \approx 5.1962}$$

22. Find the standard deviation for the probability density function with $E[X] = 2.0$.

Value of X	Probability
0	0.4
2	0.3
4	0.2
6	0.1

Solution

Complete the probability density function to find the standard deviation.

Value of x	Probability (p)	$x \cdot p$	$(x-\mu)^2 \cdot p$
0	0.4	0	$(0-2)^2 \cdot 0.4 = 1.6$
2	0.3	0.6	$(2-2)^2 \cdot 0.3 = 0$
4	0.2	0.8	$(4-2)^2 \cdot 0.2 = 0.8$
6	0.1	0.6	$(6-2)^2 \cdot 0.1 = 1.6$

$$\sigma^2 = 1.6 + 0 + 0.8 + 1.6 = 4$$
$$\sigma = 2$$

$$\boxed{\sigma = 2}$$

23. A carnival game costs \$1 to play. The probability that you win \$50 is 0.001. The probability that you win \$10 is 0.05. The probability that you win \$5 is 0.10. What is your expected return per game?

Solution

To find the expected return per game, we need to create a probability density function. All of the values that you could win need to have one dollar subtracted for the cost of the game. This makes the possible values \$49, \$9, \$4 and -\$1, if you do not win any money. To find the probability of losing \$1, subtract the other probabilities from 1.

x	$p(x)$	$x \cdot p(x)$
49	0.001	0.049
9	0.05	0.45
4	0.10	0.4
-1	0.849	-0.849

$$\mu = 0.049 + 0.45 + 0.4 - 0.849 = 0.05$$

$$\boxed{\text{The expected return per game is 5 cents.}}$$

Chapter 6: Linear Equations and Matrix Algebra

24. Solve the following system of linear equations using the substitution or elimination methods.

$$\begin{aligned} 3x + 4y &= 2 \\ 6x - 8y &= 0 \end{aligned}$$

Solution

A simple way to approach this problem is by using the elimination method. By multiplying the first equation by 2, we will cancel the y when we add the equations together.

$$\begin{aligned} 3x + 4y &= 2 \\ 2(3x + 4y) &= 2(2) \\ 6x + 8y &= 4 \end{aligned}$$

After canceling y, solve for x.

$$\begin{array}{cccc} 6x & +8y & = & 4 \\ +6x & -8y & = & 0 \\ \hline 12x & +0y & = & 4 \\ & 12x & = & 4 \\ & x & = & \dfrac{1}{3} \end{array}$$

After finding x, you can plug in your answer to either equation to solve for y.

$$\begin{aligned} 6x + 8y &= 4 \\ 6\left(\frac{1}{3}\right) + 8y &= 4 \\ 2 + 8y &= 4 \\ 8y &= 2 \\ y &= \frac{1}{4} \end{aligned}$$

$$\boxed{\left(\frac{1}{3}, \frac{1}{4}\right)}$$

25. Perform the indicated operation given the following matrices:

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & -2 \\ -2 & 0 & 1 \end{bmatrix} B = \begin{bmatrix} -1 & 0 & 6 \\ 1 & -2 & 0 \\ 0 & 1 & -3 \end{bmatrix}$$

(a) $B - 3A$

(b) AB

Solution

(a) $B - 3A$

Multiply A by 3 first, then subtract from B.

$$B - 3A = \begin{bmatrix} -1 & 0 & 6 \\ 1 & -2 & 0 \\ 0 & 1 & -3 \end{bmatrix} - \begin{bmatrix} 3 & -3 & 0 \\ 6 & 9 & -6 \\ -6 & 0 & 3 \end{bmatrix} = \begin{bmatrix} -4 & 3 & 6 \\ -5 & -11 & 6 \\ 6 & 1 & -6 \end{bmatrix}$$

(b) AB

The number of rows in A and the number of columns in B determine the number of rows and columns of the product AB. Since A and B are both 3×3 matrices, their product will also be a 3×3 matrix.

$$\begin{matrix} A & B \\ \mathbf{3} \times 3 & 3 \times \mathbf{3} \end{matrix}$$

$$AB = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & -2 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 6 \\ 1 & -2 & 0 \\ 0 & 1 & -3 \end{bmatrix} = \begin{bmatrix} -2 & 2 & 6 \\ 1 & -8 & 18 \\ 2 & 1 & -15 \end{bmatrix}$$

26. Solve the following system of linear equations using the substitution or elimination methods.

$$\begin{aligned} x + y + z &= 6 \\ 2x - y - z &= -3 \\ x - 2y + 3z &= 6 \end{aligned}$$

Solution

Here is one way to solve this problem using the elimination method. We can start by adding the first and second equations, as this will eliminate y and z.

$$\begin{array}{cccrr} x & +y & +z & = & 6 \\ 2x & -y & -z & = & -3 \\ \hline 3x & +0y & +0z & = & 3 \\ & & 3x & = & 3 \\ & & x & = & 1 \end{array}$$

Next, we can plug the value of x into two other equations.

$$\begin{aligned} 2x - y - z &= -3 \\ 2(1) - y - z &= -3 \\ 2 - y - z &= -3 \\ -y - z &= -5 \end{aligned}$$

$$\begin{aligned} x - 2y + 3z &= 6 \\ 1 - 2y + 3z &= 6 \\ -2y + 3z &= 5 \end{aligned}$$

Before adding these equations together, we can multiply the first equation by 3 that way the z will cancel.

$$\begin{aligned} 3(-y - z) &= 3(-5) \\ -3y - 3z &= -15 \end{aligned}$$

$$\begin{array}{ccrr} -3y & -3z & = & -15 \\ -2y & +3z & = & 5 \\ \hline -5y & +0z & = & -10 \\ & -5y & = & -10 \\ & y & = & 2 \end{array}$$

After finding the value of y, plug in the values of x and y into any equation to solve for z.

$$\begin{aligned} x + y + z &= 6 \\ 1 + 2 + z &= 6 \\ 3 + z &= 6 \\ z &= 3 \end{aligned}$$

$$\boxed{x = 1,\ y = 2,\ z = 3}$$

27. Find AB and BA, if possible:

$$A = \begin{bmatrix} -4 & 1 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} -4 & 2 \\ 1 & 0 \\ 6 & -9 \end{bmatrix}$$

Solution

AB is possible because the number of columns in A correspond to the number of rows in B.

$$\begin{array}{cc} A & B \\ 1 \times \mathbf{3} & \mathbf{3} \times 2 \end{array}$$

$$AB = A = \begin{bmatrix} -4 & 1 & 3 \end{bmatrix} \begin{bmatrix} -4 & 2 \\ 1 & 0 \\ 6 & -9 \end{bmatrix} = \boxed{\begin{bmatrix} 35 & -35 \end{bmatrix}}$$

BA is not possible; the number of columns in matrix B does not correspond to the number of rows in matrix A.

$$\begin{array}{cc} B & A \\ 3 \times \mathbf{2} & \mathbf{1} \times 3 \end{array}$$

28. A large state university offers two math courses: Finite Math and Applied Calculus. Each section of Finite Math has 60 students and each section of Applied Calculus has 50. The department will offer a total of 110 sections in a semester, and 6000 students would like to take a math course. How many sections of each course should the department offer in order to fill all sections and accomodate all of the students? Hint: Set up a system of equations where $x =$ the number of Finite sections and $y =$ the number of Applied Calculus sections.

Solution

First, we need to create the equations.

$$\begin{aligned} x + y &= 110 \\ 60x + 50y &= 6000 \end{aligned}$$

Next, we can use elimination or substitution to solve. Here is an example using substitution.

$$\begin{aligned} x + y &= 110 \\ x &= 110 - y \end{aligned}$$

$$\begin{aligned} 60x + 50y &= 6000 \\ 60(110 - y) + 50y &= 6000 \\ 6600 - 60y + 50y &= 6000 \\ -10y &= -600 \\ y &= 60 \end{aligned} \qquad \begin{aligned} x &= 110 - y \\ x &= 110 - 60 \\ x &= 50 \end{aligned}$$

$\boxed{\text{The department should offer 50 sections of Finite and 60 sections of Applied Calculus.}}$

29. Solve the following system of linear equations using the substitution or elimination methods.

$$\begin{aligned} x + 4y &= 12 \\ -3x - 12y &= -24 \end{aligned}$$

Solution

Here is an example of how to solve this problem using elimination. We can start by multiplying the first equation by 3, and then adding the two equations together.

$$\begin{aligned} x + 4y &= 12 \\ 3(x + 4y) &= 3(12) \\ 3x + 12y &= 36 \end{aligned}$$

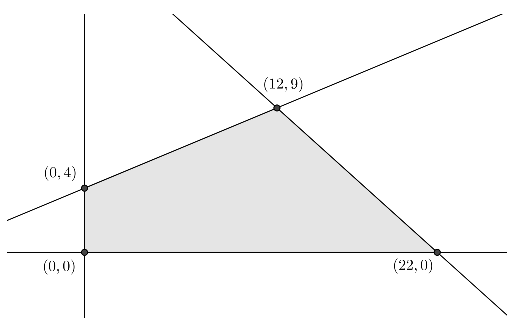
$$\begin{array}{rrcr} 3x & +12y & = & 36 \\ -3x & -12y & = & -24 \\ \hline 0x & +0y & = & 12 \\ & 0 & \neq & 12 \end{array}$$

If both x and y cancel and the resulting equation is false, it means that the lines are parallel, therefore they never intersect. This means that the lines are independent and inconsistent.

$\boxed{\text{No solution}}$

Chapter 7: Linear Programming: Graphical Solutions

30. Find the maximum value of $F = -30x + 50y$ in the feasible region shown below.



Solution

Corner Points	Function Values: $F(x, y)$
$(0, 0)$	$F(0, 0) = -30(0) + 50(0) = 0$
$(0, 4)$	$F(0, 4) = -30(0) + 50(4) = 200$
$(12, 9)$	$F(12, 9) = -30(12) + 50(9) = 90$
$(22, 0)$	$F(22, 0) = -30(22) + 50(0) = -660$

$\boxed{\text{Max} = 200 \text{ when } x = 0 \text{ and } y = 4}$

31. Graph the following system of linear inequalities and shade the feasible region. Is it bounded or unbounded? Find the coordinates of all corner points. Show all your work.

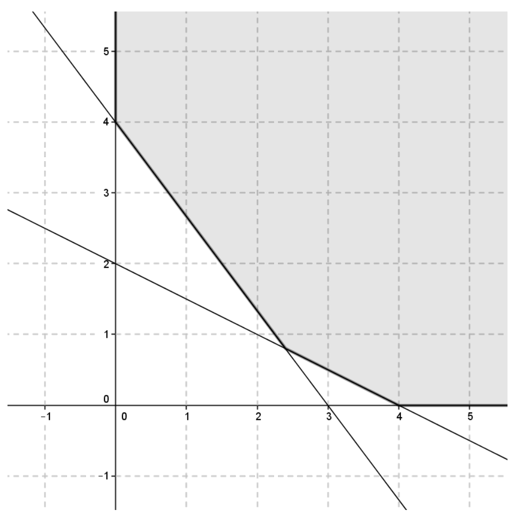
$$\begin{aligned} 2x + 4y &\geq 8 \\ 4x + 3y &\geq 12 \\ x, y &\geq 0 \end{aligned}$$

Solution

To start graphing the equations, it is typically best to either rewrite the equations in $y = mx + b$ form or solve for the x and y-intercepts. Here is an example where the equations have been rewritten in $y = mx + b$ form.

$$\begin{aligned} y &\geq -\frac{1}{2}x + 2 \\ y &\geq -\frac{4}{3}x + 4 \\ x, y &\geq 0 \end{aligned}$$

Note that the lines $x, y \geq 0$ are actually two separate lines that intersect at the origin, $(0, 0)$. If you have the constraints $x, y \geq 0$, the feasible region is limited to Quadrant 1, the upper right quadrant of the graph.



Two of the corner points can be easily identified by looking at the graph. These are the points $(0, 4)$ and $(4, 0)$. To find the third corner point, we will have to find the solution of the two intersecting linear inequalities. To do this, we need to convert the inequalities into equations.

$$\begin{aligned} 2x + 4y \geq 8 &\quad \rightarrow \quad 2x + 4y = 8 \\ 4x + 3y \geq 12 &\quad \rightarrow \quad 4x + 3y = 12 \end{aligned}$$

One way to solve these equations is by using the elimination method. By multiplying the first equation by -2, we can eliminate x when we add the equations together.

$$\begin{aligned} -2 \cdot (2x + 4y) &= -2 \cdot (8) \\ -4x - 8y &= -16 \end{aligned}$$

$$\begin{array}{llcr} -4x & -8y & = & -16 \\ 4x & +3y & = & 12 \\ \hline 0x & -5y & = & -4 \\ & -5y & = & -4 \\ & y & = & \frac{4}{5} \end{array}$$

Next, plug in the value of y to either equation to solve for x.

$$\begin{aligned} 2x + 4y &= 8 \\ 2x + 4\left(\frac{4}{5}\right) &= 8 \\ 2x + \frac{16}{5} &= 8 \\ 2x &= 8 - \frac{16}{5} \\ 2x &= \frac{40}{5} - \frac{16}{5} \\ 2x &= \frac{24}{5} \\ x &= \frac{12}{5} \end{aligned}$$

This gives us our final corner point of $(\frac{12}{5}, \frac{4}{5})$.

$$\boxed{\text{The solution is unbounded. The corner points are } (0,4),\ (4,0),\ \text{and } \left(\frac{12}{5}, \frac{4}{5}\right).}$$

32. A farm consists of 240 acres of land. The farmer wishes to plant this acreage in corn or oats. Profit per acre in corn is \$40 and that in oats is \$30. An additional restriction is that the total hours of labor during the season is 320. Each acre of land in corn requires 2 hours of labor, whereas oats require 1 hour per acre. Determine how the land should be divided between corn and oats in order to give maximum profit.

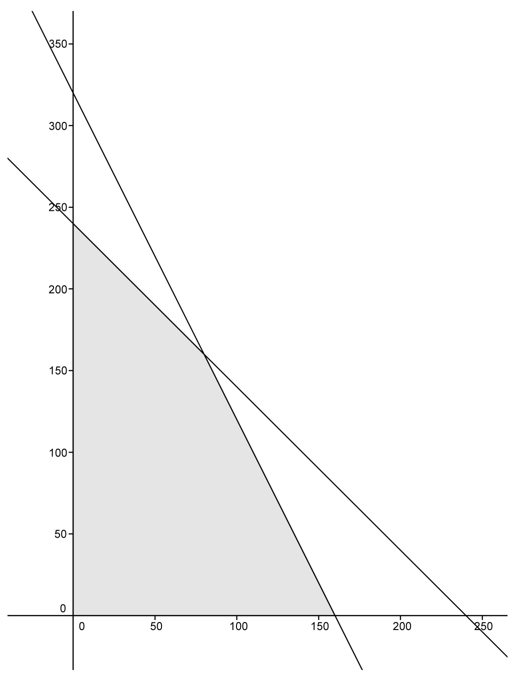
Solution

To solve this problem, we have to determine what our restraints are. Let $x =$ the number of acres of corn and $y =$ the number of acres of oats. Since the farm consists of 240 acres, x and y have to be less than or equal to 240. Also, the total hours of labor has to be less than or equal to 320. So, adding the time it takes for each acre of corn to the time it takes for each acre of oats has to be less than or equal to 320. Lastly, to find our profit maximizing equation, multiply x and y by their individual profits per acre and add them together.

$$x + y \leq 240$$
$$2x + y \leq 320$$
$$x, y \geq 0$$

$$P = 40x + 30y$$

After finding the equations, we can graph them to find the corner points of the feasible region.



The corner point $(0, 0)$ is easy to see from the graph, however the other corner points are less obvious. Now, we need to find the y-intercept of the first inequality, the x-intercept of the second inequality, and the intersection of the two inequalities.

$$\begin{aligned} x + y &= 240 \\ 0 + y &= 240 \\ y &= 240 \\ &(0, 240) \end{aligned}$$

$$\begin{aligned} 2x + y &= 320 \\ 2x + 0 &= 320 \\ x &= 160 \\ &(160, 0) \end{aligned}$$

By multiplying the first equation by -1, we can eliminate y when we add the two equations together.

$$\begin{aligned} -1 \cdot (x + y) &= -1 \cdot (240) \\ -x - y - 240 \end{aligned}$$

$$\begin{array}{cccr} -x & -y & = & -240 \\ 2x & +y & = & 320 \\ \hline x & +0y & = & 80 \\ & x & = & 80 \end{array}$$

By plugging in the value of x into one of the equations, we can solve for y, and find the last corner point.

$$\begin{aligned} x + y &= 240 \\ 80 + y &= 240 \\ y &= 160 \\ &(80, 160) \end{aligned}$$

The final step is to use the corner points to find the maximum profit.

Corner Points	Function Values: $P(x, y)$
$(0, 0)$	$P(0, 0) = 40(0) + 30(0) = 0$
$(0, 240)$	$P(0, 240) = 40(0) + 30(240) = 7200$
$(160, 0)$	$P(160, 0) = 40(160) + 30(0) = 6400$
$(80, 160)$	$P(80, 160) = 40(80) + 30(160) = 8000$

The land should be divided into 80 acres of corn and 160 acres of oats to yield a maximum profit of $8,000.

Chapter 9: Markov Chains

33. For the given transition matrix and state vector, find the state vector

$$T = \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix} \qquad P_0 = \begin{bmatrix} 0.7 & 0.3 \end{bmatrix}$$

(a) one transition later.

(b) two transitions later.

Solution

(a) one transition later.

To solve these problems, use the State Vector Formula, where n is the number of transitions.

$$\text{State Vector Formula: } P_n = P_{n-1} \cdot T \text{ or } P_n = P_o \cdot T^n$$

$$P_1 = P_o \cdot T^1$$

$$P_1 = \begin{bmatrix} 0.7 & 0.3 \end{bmatrix} \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix}$$

$$P_1 = \boxed{\begin{bmatrix} 0.48 & 0.52 \end{bmatrix}}$$

(b) two transitions later.

$$P_n = P_{n-1} \cdot T$$

$$P_2 = P_{2-1} \cdot T$$

$$P_2 = P_1 \cdot T$$

$$P_2 = \begin{bmatrix} 0.48 & 0.52 \end{bmatrix} \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix}$$

$$P_2 = \boxed{\begin{bmatrix} 0.392 & 0.608 \end{bmatrix}}$$

34. A town has two plumbers. 50% of those who call plumber A will call plumber B the next time, and 40% of those who call plumber B will call plumber A the next time. If plumber A was most recently used for a repair, what is the probability that plumber B will be used two repairs later?

Solution

To begin this problem, we need to create the transition matrix.

$$T = \begin{array}{c} \\ A \\ B \end{array} \begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \begin{bmatrix} 0.5 & 0.5 \\ 0.4 & 0.6 \end{bmatrix} \end{array}$$

Since we are finding the probability that plumber B will be used two repairs later given that plumber A was most recently used, we will need to find element t_{AB} of T^2.

$$T^2 = T \cdot T$$

$$T^2 = \begin{bmatrix} 0.5 & 0.5 \\ 0.4 & 0.6 \end{bmatrix} \cdot \begin{bmatrix} 0.5 & 0.5 \\ 0.4 & 0.6 \end{bmatrix}$$

$$T^2 = \begin{bmatrix} 0.45 & 0.55 \\ 0.44 & 0.56 \end{bmatrix}$$

$$\boxed{t_{AB} = 0.55}$$

35. For the given transition matrix and state vector, find the state vector

$$T = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad P_0 = \begin{bmatrix} 0.1 & 0.4 & 0.5 \end{bmatrix}$$

(a) one transition later.
(b) two transitions later.

Solution

(a) one transition later.

State Vector Formula: $P_n = P_{n-1} \cdot T$ or $P_n = P_o \cdot T^n$

$$P_1 = P_o \cdot T^1$$

$$P_1 = \begin{bmatrix} 0.1 & 0.4 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & 0.5 & 0.5 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P_1 = \boxed{\begin{bmatrix} 0.40 & 0.55 & 0.05 \end{bmatrix}}$$

(b) two transitions later.

$$P_n = P_{n-1} \cdot T$$

$$P_2 = P_{2-1} \cdot T$$

$$P_2 = \begin{bmatrix} 0.40 & 0.55 & 0.05 \end{bmatrix} \begin{bmatrix} 0 & 0.5 & 0.5 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$P_2 = \boxed{\begin{bmatrix} 0.55 & 0.25 & 0.20 \end{bmatrix}}$$

36. Aaron sends his mother a birthday card every year. Sometimes he sends a funny card. Sometimes he sends a sentimental card. His mother has kept track of the type sent and calculates that a funny card is followed the next year by a sentimental one 80% of the time. A sentimental card is followed by a funny one 60% of the time. If a sentimental card is sent this year, what is the probability that another sentimental card will be sent four years from now?

Solution

To begin this problem, we need to create the transition matrix.

$$T = \begin{array}{c c} & \begin{array}{cc} \text{F} & \text{S} \end{array} \\ \begin{array}{c} \text{F} \\ \text{S} \end{array} & \begin{bmatrix} 0.2 & 0.8 \\ 0.6 & 0.4 \end{bmatrix} \end{array}$$

In this problem we have two states, F and S. The probability that Aaron's card will be sentimental (S) after 4 repetitions given that a sentimental card (S) is sent this year is the t_{SS} element of T^4.

$$T^2 = T \cdot T$$

$$T^2 = \begin{bmatrix} 0.2 & 0.8 \\ 0.6 & 0.4 \end{bmatrix} \cdot \begin{bmatrix} 0.2 & 0.8 \\ 0.6 & 0.4 \end{bmatrix}$$

$$T^2 = \begin{bmatrix} 0.52 & 0.48 \\ 0.36 & 0.64 \end{bmatrix}$$

$$T^4 = T^2 \cdot T^2$$

$$T^4 = \begin{bmatrix} 0.52 & 0.48 \\ 0.36 & 0.64 \end{bmatrix} \cdot \begin{bmatrix} 0.52 & 0.48 \\ 0.36 & 0.64 \end{bmatrix}$$

$$T^4 = \begin{bmatrix} 0.4432 & 0.5568 \\ 0.4176 & 0.5824 \end{bmatrix}$$

$$\boxed{t_{SS} = 0.5824}$$

37. Taylor's and Sower's are two stores that do alterations of garments. 40% of Taylor's customers take their next garment to Sower's, and 30% of Sower's customers take their next job to Taylor's. In the long run, what will be the distribution of the market between these two stores? (i.e., find the steady state vector)

Solution

$$T = \begin{array}{c c} & \begin{array}{cc} T & S \end{array} \\ \begin{array}{c} T \\ S \end{array} & \begin{bmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{bmatrix} \end{array}$$

By multiplying the transition matrix by the matrix $\begin{bmatrix} x & y \end{bmatrix}$ and setting it equal to the matrix $\begin{bmatrix} x & y \end{bmatrix}$, we can create two equations. Solving these equations will give us the steady state vector.

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 0.6 & 0.4 \\ 0.3 & 0.7 \end{bmatrix} = \begin{bmatrix} x & y \end{bmatrix}$$

Use substitution to solve the equations below:

$$\begin{aligned} 0.6x + 0.3y &= x \\ 0.4x + 0.7y &= y \\ x + y &= 1 \rightarrow y = -x + 1 \end{aligned}$$

$$\begin{aligned} 0.6x + 0.3(-x + 1) &= x \\ 0.6x - 0.3x + 0.3 &= x \\ 0.3x + 0.3 &= x \\ 0.3 &= 0.7x \\ \frac{3}{7} &= x \end{aligned}$$

$$\begin{aligned} \frac{3}{7} + y &= 1 \\ y &= \frac{4}{7} \end{aligned}$$

Steady state vector: $\boxed{x = \frac{3}{7},\ y = \frac{4}{7}}$

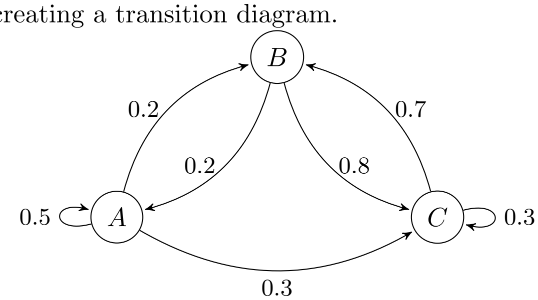
38. Determine whether the following transition matrix is regular.

$$\begin{bmatrix} 0.5 & 0.2 & 0.3 \\ 0.2 & 0 & 0.8 \\ 0 & 0.7 & 0.3 \end{bmatrix}$$

Solution

To determine whether or not a transition matrix is regular, we first have to see whether or not it is irreducible. A transition matrix is said to be irreducible if all states communicate. This means that you should be able to go from any state to the rest of the states either directly of indirectly. The best way to easily see this is by creating a transition diagram.

$$T = \begin{array}{c c} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0.5 & 0.2 & 0.3 \\ 0.2 & 0 & 0.8 \\ 0 & 0.7 & 0.3 \end{bmatrix} \end{array}$$



After we have created the transition diagram, we can see that the matrix is irreducible. Lastly, we need to determine whether the diagonal that runs from the upper left to the lower right has at least one non-zero entry.

$$\begin{bmatrix} 0.5 & & \\ & 0 & \\ & & 0.3 \end{bmatrix}$$

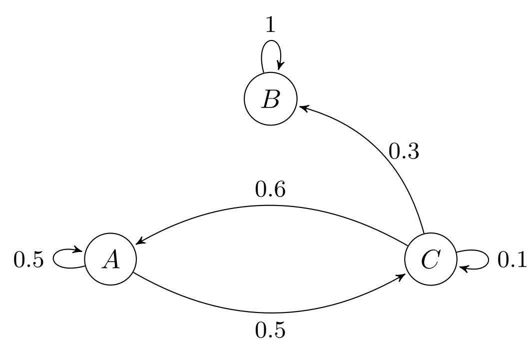
Since this matrix has two non-zero entries on the diagonal, it is $\boxed{\text{regular}}$.

39. Determine whether the following transition matrix is regular.

$$\begin{bmatrix} 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \\ 0.6 & 0.3 & 0.1 \end{bmatrix}$$

Solution

$$T = \begin{array}{c c} & \begin{array}{ccc} A & B & C \end{array} \\ \begin{array}{c} A \\ B \\ C \end{array} & \begin{bmatrix} 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \\ 0.6 & 0.3 & 0.1 \end{bmatrix} \end{array}$$



After we have created the transition diagram, we can see that the matrix is not irreducible, therefore it is $\boxed{\text{not regular}}$.