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1 Systems of Linear Equations and Matrices

1.1 Gaussian Elimination
Solve the linear system by Gaussian elimination.

\[ \begin{align*}
2x + 2y + 2z &= 0 \\
-2x + 5y + 2z &= 1 \\
8x + y + 4z &= -1
\end{align*} \]

1.2 Elementary Matrices and a Method for Finding \( A^{-1} \)
Use row operations to find the inverse matrix.

\[ \begin{bmatrix}
2 & 6 & 6 \\
2 & 7 & 6 \\
2 & 7 & 7
\end{bmatrix} \]

1.3 More on Linear Systems and Invertible Matrices
Determine the conditions on \( b \)'s, if any, in order to guarantee that the linear system is consistent.

\[ \begin{align*}
x - 2y + 5z &= b_1 \\
4x - 5y + 8z &= b_2 \\
-3x + 3y - 3z &= b_3
\end{align*} \]

1.4 Diagonal, triangular, and Symmetric Matrices
Determine, by inspection, whether the following matrices are invertible.

(a) \( \begin{bmatrix}
0 & 6 & -1 \\
0 & 7 & -4 \\
0 & 0 & -2
\end{bmatrix} \)  
(b) \( \begin{bmatrix}
-1 & 2 & 4 \\
0 & 3 & 0 \\
0 & 0 & 5
\end{bmatrix} \)

1.5 Matrix Transformations
The images of the standard basis vectors for \( \mathbb{R}^3 \) are given for a linear transformation \( T: \mathbb{R}^3 \to \mathbb{R}^3 \). find the standared matrix for the transformation and find \( T(\bar{x}) \).

\[ T(\bar{e}_1) = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \ T(\bar{e}_2) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \ T(\bar{e}_3) = \begin{bmatrix} 4 \\ -3 \\ -1 \end{bmatrix}, \ \bar{x} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} \]
2 Determinants

2.1 Determinants by Cofactor Expansion
Evaluate the determinant of the matrix by cofactor expansion.

\[
A = \begin{bmatrix}
3 & 3 & 1 \\
1 & 0 & -4 \\
1 & -3 & 5
\end{bmatrix}
\]

2.2 Evaluating Determinants by Row Reduction
Evaluate the determinant of the matrix by using row reduction.

\[
A = \begin{bmatrix}
3 & 6 & -9 \\
0 & 0 & -2 \\
-2 & 1 & 5
\end{bmatrix}
\]

2.3 Properties of Determinants; Cramer’s Rule
Find the values of k for which the matrix A is invertible.

\[
A = \begin{bmatrix}
1 & 2 & 0 \\
k & 1 & k \\
0 & 2 & 1
\end{bmatrix}
\]

3 Euclidean Vector Spaces

3.1 Vectors in 2-space, 3-space, and n-space
Let \( \mathbf{u} = \langle 1, -1, 3, 5 \rangle \) and \( \mathbf{v} = \langle 2, 1, 0, -3 \rangle \). Find scalars a and b so that

\[a\mathbf{u} + b\mathbf{v} = \langle 1, -4, 9, 18 \rangle\]

3.2 Norm, Dot Products, and Distances in \( \mathbb{R}^n \)
Find the Euclidean distance between \( \mathbf{u} \) and \( \mathbf{v} \) and the cosine of the angle between those vectors. State whether the angle is acute, obtuse, or right.

\[\mathbf{u} = \langle 3, 3, 3 \rangle \]
\[\mathbf{v} = \langle 1, 0, 4 \rangle \]

3.3 Orthogonality
Determine whether the given planes are perpendicular.

\[3x - y + z - 4 = 0\]
\[x + 2z = -1\]
3.4 The Geometry of Linear Systems

Consider the linear systems $A\vec{x} = \vec{0}$ and $A\vec{x} = \vec{b}$

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 6 & 4 & -2 \\ -3 & -2 & 1 \end{bmatrix}, \quad \vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \text{and} \quad \vec{b} = \begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix}$$

(a) Find a solution of the homogeneous system.

(b) Confirm that $x = 1$, $x_2 = 0$, $x_3 = 1$ is a solution of the non-homogeneous system.

(c) Use parts (a) and (b) to write a general solution of the non-homogeneous system.

4 General Vector Spaces

4.1 Real Vector Spaces

Prove that the set $M_{mn}$ of all $m \times n$ matrices with the usual operations of addition and scalar multiplication is a vector space.

4.2 Subspaces

Express $\langle -9, -7, -15 \rangle$ as a linear combination of

$$\vec{u} = \langle 2, 1, 4 \rangle$$
$$\vec{v} = \langle 1, -1, 3 \rangle$$
$$\vec{w} = \langle 3, 2, 5 \rangle$$

Let $\vec{f} = \cos^2 x$ and $\vec{g} = \sin x$. Which of the following lie in the space spanned by $\vec{f}$ and $\vec{g}$?

(a) $\cos 2x$

(b) $3 + x^2$

(c) $1$

(d) $\sin x$

(e) $0$

Linear Independence

For which values of $\lambda$ do the following vectors form a linearly dependent set in $\mathbb{R}^3$?

$$\vec{V}_1 = \langle \lambda, -\frac{1}{2}, -\frac{1}{2} \rangle$$
$$\vec{V}_2 = \langle -\frac{1}{2}, \lambda - \frac{1}{2} \rangle$$
$$\vec{V}_3 = \langle -\frac{1}{2}, -\frac{1}{2}, \lambda \rangle$$
4.3 Coordinates and Basis

Is the set \( \{ (2, -3, 1), (4, 1, 1), (9, -7, 17) \} \) a basis set for \( \mathbb{R}^3 \)?

4.4 Dimension

Find a basis for the subspace of \( \mathbb{R}^4 \) that is spanned by

\[
\vec{v}_1 = \langle 1, 1, 1, 1 \rangle \\
\vec{v}_2 = \langle 2, 2, 2, 0 \rangle \\
\vec{v}_3 = \langle 0, 0, 0, 3 \rangle \\
\vec{v}_4 = \langle 3, 3, 3, 4 \rangle
\]

What is the dimension of this subspace?

4.5 Change of Basis

Let \( S \) be the standard basis for \( \mathbb{R}^2 \) and let \( B = \{ \vec{v}_1, \vec{v}_2 \} \) for \( \vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \) and \( \vec{v}_2 = \begin{bmatrix} -3 \\ 4 \end{bmatrix} \). Find the transition matrix \( P_{s \rightarrow B} \) and use it to find \( [\vec{w}_B] \) for \( \vec{w} = \begin{bmatrix} 5 \\ -3 \end{bmatrix} \).

4.6 Row Space, Column Space, and Null Space

Find a basis for the subspace of \( \mathbb{R}^4 \) that is spanned by

\[
\vec{v}_1 = \langle 1, 1, -4, -3 \rangle \\
\vec{v}_2 = \langle 3, 2, -7, 5 \rangle \\
\vec{v}_3 = \langle 2, 0, 2, -2 \rangle
\]

4.7 Rank, Nullity and the Fundamental Matrix Spaces

Given a matrix \( A \) and its reduced row echelon form matrix \( R \):

\[
A = \begin{bmatrix}
1 & 1 & 1 & 6 & 6 \\
4 & -8 & 4 & 0 & 0 \\
1 & 2 & 0 & 5 & 7 \\
1 & 0 & -2 & -5 & 1
\end{bmatrix} \quad R = \begin{bmatrix}
1 & 0 & 0 & 1 & 3 \\
0 & 1 & 0 & 2 & 2 \\
0 & 0 & 1 & 3 & 1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
(a) Deduce the rank and nullity of A
(b) Confirm that matrix A satisfies the rank-nullity theorem
(c) Write the number of leading variables and number of parameters to the system $A\vec{x} = \vec{0}$ without solving the system

4.8 Basic Matrix Transformations in $\mathbb{R}^2$ and $\mathbb{R}^3$.
Find the standard matrix for the reflection of $\mathbb{R}^2$ about the stated line. Then, use the matrix to compute the reflections about the line of the given points.
Line: makes an angle of $\frac{\pi}{3}$ with the positive x-axis
Point: $(3, 4)$ and $(\sqrt{3}, 3)$

4.9 Properties of Matrix Transformations
Find the standard for the following composition of transformations.
A rotation of 30 deg about the x-axis, followed by a rotation of 30 deg about the z-axis, followed by a contraction with a factor $k = \frac{1}{4}$

5 Eigenvalues and Eigenvectors
5.1 Eigenvalues and Eigenvectors
Find the characteristic equation, the eigenvalues, and a basis for the e-space of the matrix

$$A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

5.2 Diagonalization
For the matrix A given below,
(a) Find the geometric and algebraic multiplicity of each eigenvalue
(b) Determine if A is diagonalizable
(c) If A is diagonalizable, find a matrix P that diagonalizes it and find $P^{-1}AP$

$$A = \begin{bmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \end{bmatrix}$$

5.3 Differential Equations
Solve the System

$$\begin{align*}
y_1' &= y_1 + 3y_2 \\
y_2' &= 4y_1 + 5y_2
\end{align*}$$
Find the solution that satisfies the conditions \( y_1(0) = 2 \) and \( y_2(0) = 1 \).

5.4 Dynamical Systems and Markov Chains

Verify that the matrix \( P \) is a stochastic matrix and find the steady-state vector for the associated Markov Chain.

\[
P = \begin{bmatrix}
  1/3 & 1/4 & 2/5 \\
  0 & 3/4 & 2/5 \\
  2/3 & 0 & 1/5 
\end{bmatrix}
\]

6 Inner Product Spaces

6.1 Gram-Schmidt Process; QR-Decomposition

Let \( \mathbb{R}^3 \) have the Euclidean inner product. Find an orthonormal basis for the subspace spanned by the following vectors.

\[
\bar{V}_1 = \langle 0, 1, 2 \rangle \\
\bar{V}_2 = \langle -1, 0, 1 \rangle \\
\bar{V}_3 = \langle -1, 1, 3 \rangle
\]

6.2 Inner Product Spaces Continued

Find the QR-decomposition of the matrix

\[
A = \begin{bmatrix}
  -1 & 0 & 1 \\
  -1 & 1 & 1 \\
  1 & 0 & 1 \\
  -1 & 1 & 1 
\end{bmatrix}
\]

6.3 Best Approximation; Least Squares

Find the least squares solution to \( A\bar{x} = \bar{b} \) for the given \( A \) and \( \bar{b} \)

\[
A = \begin{bmatrix}
  2 & -2 \\
  1 & 1 \\
  3 & 1 
\end{bmatrix} \quad \bar{b} = \begin{bmatrix}
  2 \\
  -1 \\
  1 
\end{bmatrix}
\]

7 Diagonalization and Quadratic Forms

7.1 Orthogonal Matrices

Determine if the following matrices are orthogonal. If so, find its inverse.

\[
(a) \begin{bmatrix}
  1/\sqrt{2} & -1/\sqrt{2} \\
  1/\sqrt{2} & 1/\sqrt{2} 
\end{bmatrix} \quad (b) \begin{bmatrix}
  0 & 1 & 1/\sqrt{2} \\
  1 & 0 & 0 \\
  0 & 0 & 1/\sqrt{2} 
\end{bmatrix}
\]
7.2 Orthogonal Diagonalization

Find a matrix $P$ that orthogonally diagonalizes $A$ and find $P^{-1}AP$.

\[
A = \begin{bmatrix}
1 & 1 & 0 \\
1 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

7.3 Quadratic Forms

Identify the conic section represented by the equation by rotating axes to places the conic in standard position. Find an equation of the conic in rotated coordinates, and find the angle of rotation.

\[
5x^2 + 4xy + 5y^2 = 9
\]

7.4 Optimization Using Quadratic Forms

Find the maximum and minimum values of the given quadratic form subject to the given constraint.

\[
2x^2 + y^2 + z^2 + 2xy + 2xz; \quad x^2 + y^2 + z^2 = 1
\]