Contents

1 First-Order Differential Equations

1.1 Separation of Variables

Find the general solution to the given differential equations

a. $\frac{dy}{dt} = \frac{1}{2y+1}$

b. $\frac{dy}{dt} = (y^2+1)t$

c. $\frac{dy}{dt} = \frac{1}{ty+t+y+1}$

Solution

a. Move the y-terms to the left side and x-terms to the right side, then integrate.

$$\begin{aligned} \frac{dy}{dt} &= \frac{1}{2y+1} \\ (2y+1)\,dy &= dt \\ \int (2y+1)\,dy &= \int dt \\ y^2 + y &= t + C \end{aligned}$$

b. Move the y-terms to the left side and x-terms to the right side, then integrate.

$$\begin{aligned} \frac{dy}{dt} &= (y^2+1)t \\ \frac{dy}{y^2+1} &= t\,dt \\ \int \frac{dy}{y^2+1} &= \int t\,dt \\ \tan^{-1} y &= \frac{1}{2}t^2 + C \end{aligned}$$

c. Simplify the expression on the right, then separate variables and integrate as before.

$$\begin{aligned} \frac{dy}{dt} &= \frac{1}{ty+t+y+1} \\ \frac{dy}{dt} &= \frac{1}{(t+1)(y+1)} \\ (y+1)\,dy &= \frac{dt}{t+1} \\ \int (y+1)\,dy &= \int \frac{dt}{t+1} \\ \frac{1}{2}y^2 + y &= \ln|t+1| + C \end{aligned}$$

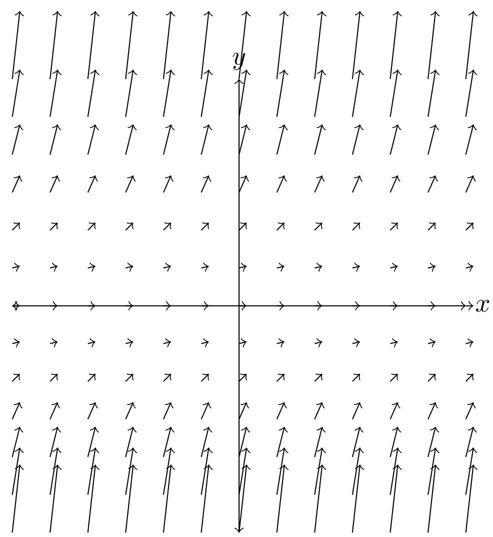
1.2 Slope Fields

Choose points (t, y) with $-2 \leq t \leq 2$ and $-2 \leq y \leq 2$ and plot part of the slope field determined by the differential equation without the use of technology.

$$\frac{dy}{dt} = 4y^2$$

Solution

The slope field itself looks as follows.



Your chosen points should align with the points from this slope field.

1.3 Equilibrium Solutions and Phase Lines

Given the differential equation

$$\frac{dy}{dt} = y^2 - 4y - 12,$$

sketch the graphs of the solutions satisfying the following initial conditions.

a. $y(0) = 1$

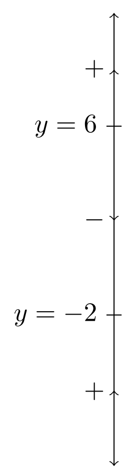
b. $y(0) = 6$

c. $y(2) = 5$

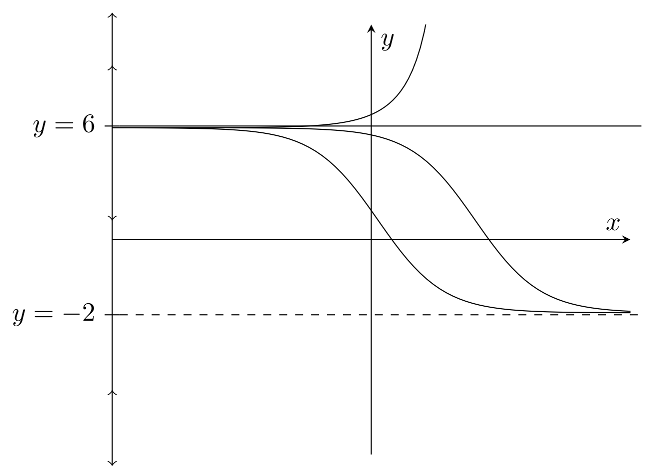
d. $y(1) = 7$

Solution

Begin by drawing the phase line for the differential equation. The only equilibrium solutions are $y = 6$ and $y = -2$, by solving the quadratic $y^2 - 4y - 12 = 0$. We then check whether the derivative is positive or negative before and after these points, like a sign chart.



We then use this, and the fact that solutions cannot cross one another to draw our solutions.



1.4 Bifurcations

Locate bifurcation values for the one-parameter family of differential equations and draw phase lines to illustrate your results.

$$\frac{dy}{dt} = (y^2 - \alpha)(y^2 - 4)$$

Solution

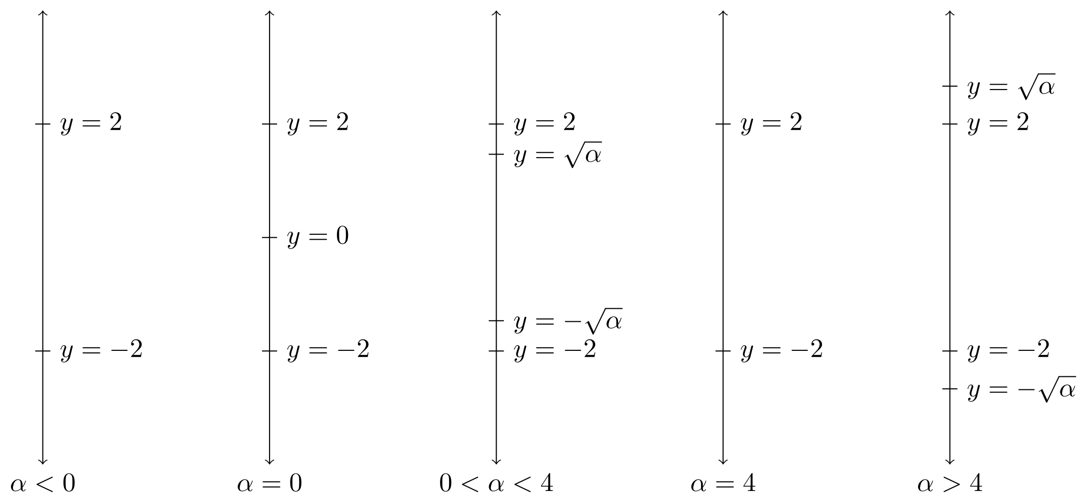
Writing $f(y) = (y^2 - \alpha)(y^2 - 4)$, one way of finding potential bifurcation values is to solve the following system of equations.

$$f(y) = 0$$
$$\frac{\partial f}{\partial y} = 0$$

For our equation, this becomes the following after simplifying the derivative.

$$(y^2 - \alpha)(y^2 - 4) = 0$$
$$2y(2y^2 - \alpha - 4) = 0$$

Solving the first equation, we get $y = \pm\sqrt{\alpha}$ or $y = \pm 2$. Plugging these into our second equation, we get $\alpha = 0$ or $\alpha = 4$. To determine whether these values are bifurcation values, we use phase lines.



From this, we see that $\alpha = 0$ is a bifurcation value since for $\alpha < 0$, there are two equilibrium solutions and for $0 < \alpha < 4$, there are four equilibrium solutions. This change in the number of equlibrium solutions is a bifurcation at $\alpha = 0$. Similarly, the number of equilibrium solutions changes at $\alpha = 4$, so we have a bifurcation there as well. Since these are the only places that the number of equilibrium solutions changes, these are the only bifurcation values.

1.5 Linear Differential Equations

Find the general solution to the given differential equations.

a. $\dfrac{dy}{dt} = -4y + 9e^{-t}$

b. $\dfrac{dy}{dt} = \dfrac{1}{2}y + 4e^{t/2}$

Solution

a. First solve for the homogeneous solution.

$$\begin{aligned} \frac{dy}{dt} &= -4y \\ \frac{dy}{y} &= -4\,dt \\ \int \frac{dy}{y} &= \int -4\,dt \\ \ln|y| &= -4t + C \\ y_h(t) &= Ce^{-4t} \end{aligned}$$

Solve for the particular solution using the method of undetermined coefficients with a guess of $y_p(t) = Ae^{-t}$.

$$\begin{aligned} -Ae^{-t} &= -4Ae^{-t} + 9e^{-t} \\ -A &= -4A + 9 \\ 3A &= 9 \\ A &= 3 \end{aligned}$$

Therefore, our general solution is as follows.

$$y_g(t) = y_p(t) + y_h(t) = 3e^{-t} + Ce^{-4t}$$

b. As before, first solve for the homogeneous solution.

$$\begin{aligned} \frac{dy}{dt} &= \frac{1}{2}y \\ \frac{dy}{y} &= \frac{1}{2}\,dt \\ \int \frac{dy}{y} &= \int \frac{1}{2}\,dt \\ \ln|y| &= \frac{1}{2}t + C \\ y_h(t) &= Ce^{t/2} \end{aligned}$$

Solve for the particular solution using the method of undetermined coefficients with a guess of $y_p(t) = Ate^{t/2}$. Note, this is actually a second guess. If you try the usual $y_p(t) = Ae^{t/2}$, all of the terms containing an A cancel out! We modify our guess by multiplying by t.

$$\begin{aligned} Ae^{t/2} + \frac{1}{2}Ate^{t/2} &= \frac{1}{2}Ate^{t/2} + 4e^{t/2} \\ Ae^{t/2} &= 4e^{t/2} \\ A &= 4 \end{aligned}$$

We now write our general solution as follows.

$$y_g(t) = y_p(t) + y_h(t) = 4te^{t/2} + Ce^{t/2}$$

1.6 Integrating Factors

Solve the given initial-value problem.

$$t\frac{dy}{dt} - 2y = 2t^3, \ y(-2) = 4$$

Solution

Since there are non-constant coefficients, we use integrating factors. First, we normalize the equation so that there is a constant 1 in front of $\frac{dy}{dt}$.

$$\frac{dy}{dt} - \frac{2}{t}y = 2t^2$$

Now, our integrating factor is given by the following formula.

$$\mu(t) = e^{\int -\frac{2}{t}\,dt} = e^{-2\ln|t|} = e^{\ln|t^{-2}|} = t^{-2}$$

We now multiply this to both sides of our equation.

$$\frac{1}{t^2}\frac{dy}{dt} - \frac{2}{t^3}y = 2$$

The reason for doing this is that now, the left hand side of the equation is (and always will be) the derivative of $y(t)$ times our integrating factor, $1/t^2$.

$$\frac{d}{dx}\left(y(t)\frac{1}{t^2}\right) = 2$$

Now, we integrate and solve for $y(t)$.

$$\begin{aligned} \int \frac{d}{dx}\left(y(t)\frac{1}{t^2}\right)dt &= \int 2\,dt \\ y(t)\frac{1}{t^2} &= 2t + C \\ y(t) &= 2t^3 + Ct^2 \end{aligned}$$

Now that we have a general solution, we plug in our initial-conditions to solve for C.

$$\begin{aligned} 4 &= 2(-2)^3 + C(-2)^2 \\ 4 &= 16 + 4C \\ 4C &= -12 \\ C &= -3 \end{aligned}$$

Then, we arrive at our solution.

$$y(t) = 2t^3 - 3t^2$$

2 Systems of Differential Equations

2.1 Damped Harmonic Oscillators

Use the method of undetermined coefficients to find two non-zero solutions that are not multiples of one another.

$$\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 10y = 0$$

Solution

With constant coefficients, the standard guess is $y(t) = e^{rt}$. Plugging this into our equation, we get

$$\begin{aligned} r^2e^{rt} + 7re^{rt} + 10e^{rt} &= 0 \\ r^2 + 7r + 10 &= 0 \\ (r+2)(r+5) &= 0 \\ r = -2 \quad \text{or} \quad r &= -5 \end{aligned}$$

Now we have two, non-zero solutions that are not multiples of one another, $y_1(t) = e^{-2t}$ and $y_2(t) = e^{-5t}$.

2.2 Decoupled Systems

Consider the following system of differential equations.

$$\frac{dy}{dt} = 2x - 8y^2$$
$$\frac{dy}{dt} = -3y$$

Derive the general solution and find the solution that corresponds to the initial condition $(x_0, y_0) = (0, 1)$.

Solution

Decoupled systems are special in that, they can be easily solved piece by piece. We first solve the second equation by separation of variables.

$$\frac{dx}{dt} = -3y$$
$$\frac{dy}{y} = -3\,dt$$
$$\int \frac{dy}{y} = \int -3\,dt$$
$$\ln|y| = -3t + C$$
$$y = C_1 e^{-3t}$$

We now take this, and plug it into our equation for x and get a linear equation that we know how to solve.

$$\frac{dx}{dt} = 2x - 8C_1^2 e^{-6t}$$

The solution with either undetermined coefficients or integrating factors comes out to be

$$x = C_1^2 e^{-6t} + C_2 e^{2t}$$

We then have to solve for C_1 and C_2 using our initial condition $(x_0, y_0) = (0, 1)$. Plugging in $t = 0$, we get

$$C_1 = 1,\ C_1^2 + C_2 = 0 \implies C_1 = 1,\ C_2 = -1.$$

We then have our solution.

$$x(t) = e^{-6t} - e^{-2t}$$
$$y(t) = e^{-3t}$$

2.3 Systems of Linear Equations

For the given system of linear equations

$$\frac{dx}{dt} = -2x - y$$
$$\frac{dy}{dt} = 2x - 5y,$$

form the matrix equation that represents it and check that the functions $\vec{Y}_1(t)$ and $\vec{Y}_2(t)$ are solutions. If the two solutions are linearly independent, form the general solution.

$$\vec{Y}_1(t) = \langle e^{-3t} - 2e^{-4t}, e^{-3t} - 4e^{-4t} \rangle$$
$$\vec{Y}_2(t) = \langle 2e^{-3t} + e^{-4t}, 2e^{-3t} + 2e^{-4t} \rangle$$

Solution

The matrix equation can be easily written by taking the coefficients of the x-terms and y-terms.

$$\frac{d\vec{Y}}{dt} = \begin{pmatrix} -2 & -1 \\ 2 & -5 \end{pmatrix} \vec{Y}$$

To check if $\vec{Y}_1(t)$ is a solution, we simply see if it satisfies the equation above.

$$\frac{d\vec{Y}_1}{dt} = \begin{pmatrix} -3e^{-3t} + 8e^{-4t} \\ -3e^{-3t} + 16e^{-4t} \end{pmatrix}$$
$$\begin{pmatrix} -2 & -1 \\ 2 & -5 \end{pmatrix} \vec{Y}_1(t) = \begin{pmatrix} -2(e^{-3t} - 2e^{-4t}) - (e^{-3t} - 4e^{-4t}) \\ 2(e^{-3t} - 2e^{-4t}) - 5(e^{-3t} - 4e^{-4t}) \end{pmatrix} = \begin{pmatrix} -3e^{-3t} + 8e^{-4t} \\ -3e^{-3t} + 16e^{-4t} \end{pmatrix}$$

Since the two are equal, we have that $\vec{Y}_1(t)$ is a solution. Similarly, we check if $\vec{Y}_2(t)$ is a solution.

$$\frac{d\vec{Y}_2}{dt} = \begin{pmatrix} -6e^{-3t} - 4e^{-4t} \\ -6e^{-3t} - 8e^{-4t} \end{pmatrix}$$
$$\begin{pmatrix} -2 & -1 \\ 2 & -5 \end{pmatrix} \vec{Y}_2(t) = \begin{pmatrix} -2(2e^{-3t} + e^{-4t}) - (2e^{-3t} + 2e^{-4t}) \\ 2(2e^{-3t} + e^{-4t}) - 5(2e^{-3t} + 2e^{-4t}) \end{pmatrix} = \begin{pmatrix} -6e^{-3t} - 4e^{-4t} \\ -6e^{-3t} - 8e^{-4t} \end{pmatrix}$$

As before, since the two are equal, $\vec{Y}_2(t)$ is a solution. Now we must check that the solutions are linearly independent. To do so, we only need to check whether or not the solutions are multiples of one another (since there are only 2, if there are more solutions, this method cannot be used!). Note that if we plug in $t = 0$, we have

$$\vec{Y}_1(0) = \begin{pmatrix} -1 \\ -3 \end{pmatrix}, \ \vec{Y}_2(0) = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

Since these vectors are not multiples of one another, $\vec{Y}_1(t)$ and $\vec{Y}_2(t)$ cannot be multiples of one another and hence, are linearly independent. Since they are linearly independent, we may write the general solution as follows.

$$\vec{Y}_g(t) = c_1\vec{Y}_1(t) + c_2\vec{Y}_2(t) = c_1 \begin{pmatrix} e^{-3t} - 2e^{-4t} \\ e^{-3t} - 4e^{-4t} \end{pmatrix} + c_2 \begin{pmatrix} 2e^{-3t} + e^{-4t} \\ 2e^{-3t} + 2e^{-4t} \end{pmatrix}$$

2.4 Straight-Line Solutions

Solve the initial-value problem.

$$\frac{d\vec{Y}}{dt} = \begin{pmatrix} -4 & 1 \\ 2 & -3 \end{pmatrix} \vec{Y},\ \vec{Y}(0) = \langle 2, 1 \rangle$$

Solution

First, we need to find the straight-line solutions by computing the eigenvalues and eigenvectors of the matrix. The characteristic equation is computed as follows.

$$\det \begin{pmatrix} -4-\lambda & 1 \\ 2 & -3-\lambda \end{pmatrix} = \lambda^2 + 7\lambda + 10 = 0 \implies \lambda_1 = -2,\ \lambda_2 = -5$$

Now we compute our eigenvectors by finding solutions $\langle x, y \rangle$ to the following equations for $\lambda = -2$ and $\lambda = -5$ respectively.

$$\begin{pmatrix} -4-(-2) & 1 \\ 2 & -3-(-2) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies y = 2x$$
$$\begin{pmatrix} -4-(-5) & 1 \\ 2 & -3-(-5) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies y = -x$$

Using the equation $y = 2x$, we choose our first eigenvector (corresponding to $\lambda_1 = -2$) to be $\vec{v}_1 = \langle 1, 2 \rangle$. Similarly, using the equation $y = -x$, we choose our second eigenvector (corresponding to $\lambda_2 = -5$) to be $\vec{v}_2 = \langle 1, -1 \rangle$. Now we may write our general solution as follows.

$$\vec{Y}_g(t) = C_1 e^{\lambda_1 t} \vec{v}_1 + C_2 e^{\lambda_2 t} \vec{v}_2 = C_1 e^{-2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 e^{-5t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

To solve the initial-value problem, we now plug in $t = 0$ and our initial-condition to get

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} C_1 + C_2 \\ 2C_1 - C_2 \end{pmatrix}$$

So we arrive at the system of equations

$$C_1 + C_2 = 2$$
$$2C_1 - C_2 = 1.$$

This can be solved to get $C_1 = 1$ and $C_2 = 1$. Our solution to the initial-value problem is then

$$\vec{Y}(t) = e^{-2t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + e^{-5t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

2.5 Phase Portraits

Sketch the solution curves in the phase plane and the $x(t)$ and $y(t)$ graphs corresponding to the initial-value problem.

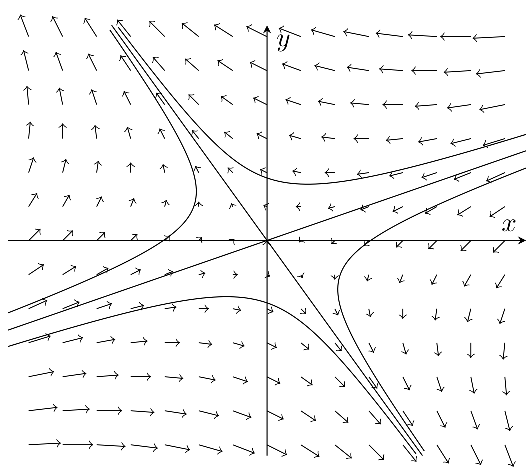
$$\frac{d\vec{Y}}{dt} = \begin{pmatrix} -2 & -2 \\ -2 & 1 \end{pmatrix} \vec{Y}, \ \vec{Y}(0) = \langle 3, -1 \rangle$$

Solution

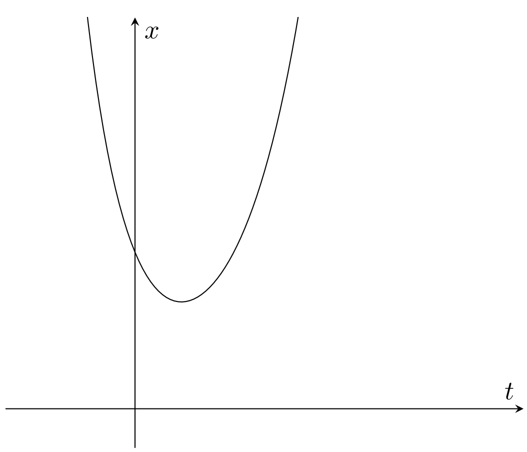
First, we need compute the eigenvalues and eigenvectors of this matrix. The characteristic equation becomes

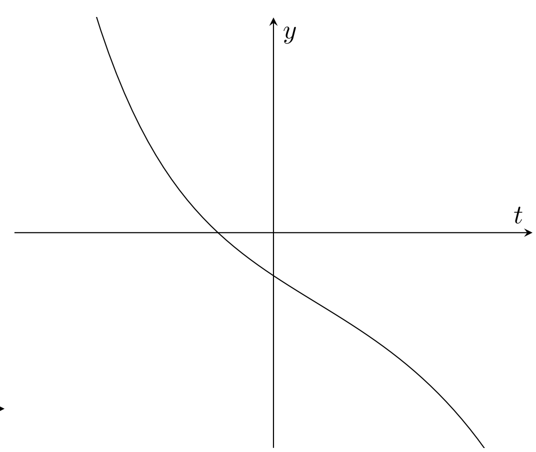
$$\det \begin{pmatrix} -2-\lambda & -2 \\ -2 & 1-\lambda \end{pmatrix} = \lambda^2 + \lambda - 6 = (\lambda - 2)(\lambda + 3) = 0$$

Which implies we have eigenvalues of $\lambda_1 = 2$ and $\lambda_2 = -3$. We compute eigenvectors of $\vec{v}_1 = \langle 1, -2 \rangle$ and $\vec{v}_2 = \langle 2, 1 \rangle$ corresponding to $\lambda_1 = 2$ and $\lambda_2 = -3$ respectively (Note: you may have multiples of our eigenvectors, this is not a problem). We now draw our phase portrait (we include the slope field to get a better visual).



The solution corresponding to the initial condition $\vec{Y}(0) = \langle 3, -1 \rangle$ is drawn going through the first and fourth quadrant above. From this, we can follow the curve to draw the $x(t)$ and $y(t)$ graphs.





2.6 Complex Eigenvalues

Consider the system below.

$$\frac{d\vec{Y}}{dt} = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix} \vec{Y}, \ \vec{Y}(0) = \langle 2, 1 \rangle$$

a. Compute the eigenvalues.

b. Determine if the origin is a spiral sink, spiral souce, or center.

c. Determine the natural period and the natural frequency.

d. Determine the direction of the oscillations in the phase plane.

e. Write the general solution and a solution to the initial-value problem.

f. Draw the solution in the phase plane.

Solution

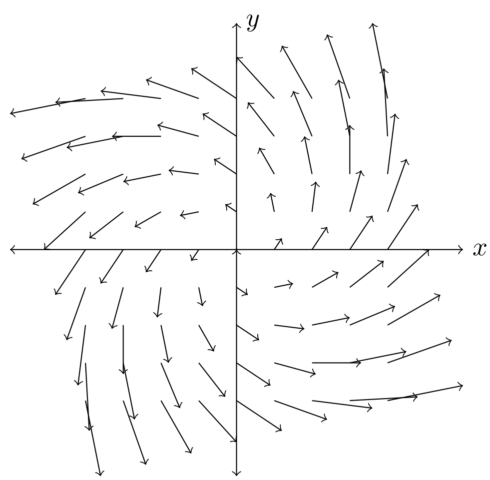
a. To compute the eigenvalues, we solve the following equation.

$$\det \begin{pmatrix} 2-\lambda & -3 \\ 3 & 2-\lambda \end{pmatrix} = \lambda^2 - 4\lambda + 13 = 0 \implies \lambda = 2 \pm 3i$$

b. To determine if the origin is a spiral sink, spiral source, or center, we look at the real part of the eigenvalues. Since the real part is $2 > 0$, we have a spiral source.

c. To find the natural period, we use the imaginary part of the eigenvalues. The period is given by $2\pi/(3) = 2\pi/3$. The natural frequency is then given by one divided by the period, which is then $3/2\pi$.

d. There are many ways to determine the direction of oscillations. Our choice is to draw part of the slope field in the phase plane. Often times, only one or two sample points may be necessary to determine the direction of oscillation.



From this, we determine the direction of oscillations is counter-clockwise.

e. To compute the general solution, we only need one straight-line solution. The straight-line solution then has a real part and an imaginary part, both of which are solutions individually. Since we already have the eigenvalues, we only need one eigenvector for a straight-line solution. We choose to find the eigenvector corresponding to $\lambda = 2 + 3i$. To solve for our eigenvector, we solve the following equation.

$$\begin{pmatrix} 2-\lambda & -3 \\ 3 & 2-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3i & -3 \\ 3 & -3i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We solve this and get $\langle x, y \rangle = \langle i, 1 \rangle$ (remember, any multiple of this vector works). We can now write our straight-line solution and solve for its real and imaginary parts.

$$e^{(2+3i)t} \begin{pmatrix} i \\ 1 \end{pmatrix} = e^{2t}(\cos 3t + i \sin 3t) \begin{pmatrix} i \\ 1 \end{pmatrix} = e^{2t} \begin{pmatrix} i\cos 3t - \sin 3t \\ \cos 3t + i \sin 3t \end{pmatrix} = e^{2t} \begin{pmatrix} -\sin 3t \\ \cos 3t \end{pmatrix} + ie^{2t} \begin{pmatrix} \cos 3t \\ \sin 3t \end{pmatrix}$$

Now both of these are solutions, and are independent! (they are not multiples of one another) Therefore, we can write our general solution as follows.

$$\vec{Y}(t) = C_1 e^{2t} \begin{pmatrix} -\sin 3t \\ \cos 3t \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} \cos 3t \\ \sin 3t \end{pmatrix}$$

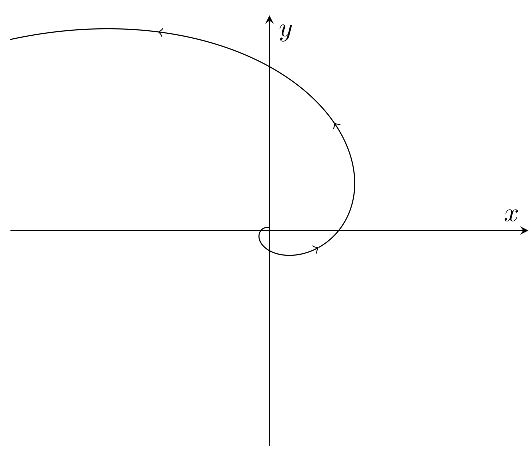
To solve the initial-value problem, we just plug in $t = 0$ to solve for C_1 and C_2.

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} = C_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \implies C_1 = 2,\ C_2 = 1$$

Therefore, our solution to the initial-value problem is

$$\vec{Y}(t) = 2e^{2t} \begin{pmatrix} -\sin 3t \\ \cos 3t \end{pmatrix} + e^{2t} \begin{pmatrix} \cos 3t \\ \sin 3t \end{pmatrix}$$

f. With the slope field from earlier, we draw the solution curve by simply following the arrows from $(2, 1)$.



2.7 Special Cases

Consider the system below.

$$\frac{d\vec{Y}}{dt} = \begin{pmatrix} -2 & -1 \\ 1 & -4 \end{pmatrix} \vec{Y}, \ \vec{Y}(0) = \langle 1, 0 \rangle$$

a. Compute the eigenvalues and associated eigenvectors.

b. Write the general solution and the solution to the initial-value problem.

c. Draw solutions in the phase plane.

Solution

a. To compute the eigenvalues, we solve the characteristic equation.

$$\det \begin{pmatrix} -2-\lambda & -1 \\ 1 & -4-\lambda \end{pmatrix} = \lambda^2 + 6\lambda + 9 = (\lambda+3)^2 = 0 \implies \lambda = -3$$

To compute the eigenvector(s), we look for solutions to the equation

$$\begin{pmatrix} -2-\lambda & -1 \\ 1 & -4-\lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Solving this system, we see the only solutions are multiples of $\langle 1, 1 \rangle$. We choose this to be our eigenvector, $\vec{v} = \langle 1, 1 \rangle$.

b. Since we have the repeated eigenvalue case, the general solution has the form

$$\vec{Y}(t) = e^{\lambda t} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + te^{\lambda t} (\mathbf{A} - \lambda \mathbf{I}) \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

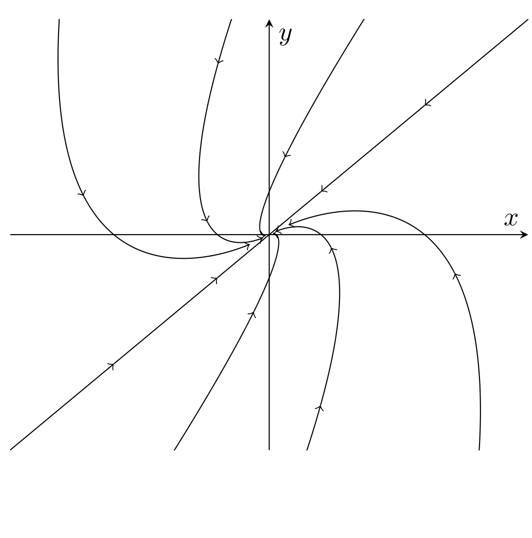
where $\mathbf{A}$ is the original matrix given in the problem. Plugging in our eigenvalue and the matrix $\mathbf{A}$, our general solution is as follows.

$$\vec{Y}(t) = e^{-3t} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + te^{-3t} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = e^{-3t} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + te^{-3t} \begin{pmatrix} x_0 - y_0 \\ x_0 - y_0 \end{pmatrix}$$

To solve the initial-value problem, we need only plug in our initial condition, $\langle x_0, y_0 \rangle = \langle 1, 0 \rangle$.

$$\vec{Y}(t) = e^{-3t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + te^{-3t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

c. The phase portrait is as follows.



Consider the system below.

$$\frac{d\vec{Y}}{dt} = \begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix} \vec{Y}, \ \vec{Y}(0) = \langle 1, 0 \rangle$$

a. Compute the eigenvalues and associated eigenvectors.

b. Write the general solution and the solution to the initial-value problem.

c. Draw solutions in the phase plane.

Solution

a. We compute the eigenvalues as normal.

$$\det \begin{pmatrix} 2-\lambda & 4 \\ 3 & 6-\lambda \end{pmatrix} = \lambda^2 - 8\lambda = \lambda(\lambda - 8) = 0 \implies \lambda_1 = 0, \ \lambda_2 = 8$$

This is considered a special case since one of our eigenvalues is zero. We compute our eigenvectors for $\lambda_1 = 0$ and $\lambda_2 = 8$ respectively as normal.

$$\lambda_1 : \begin{pmatrix} 2-\lambda_1 & 4 \\ 3 & 6-\lambda_1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies x = 2y, \ \vec{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\lambda_2 : \begin{pmatrix} 2-\lambda_2 & 4 \\ 3 & 6-\lambda_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -6 & 4 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies 3x = 2y, \ \vec{v}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Note that any multiples of our eigenvectors will still suffice as eigenvectors.

b. The general solution can be written as normal.

$$\vec{Y}(t) = C_1 e^{\lambda_1 t} \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{\lambda_2 t} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = C_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 e^{-8t} \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

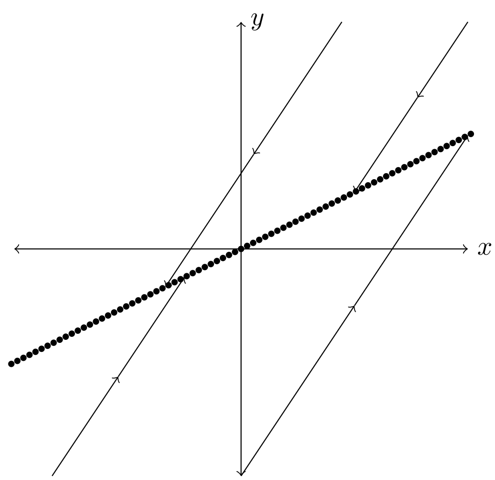
To solve the initial-value problem, we plug in $t = 0$ and our initial condition to solve for C_1 and C_2.

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = C_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 2 \\ 3 \end{pmatrix} \implies C_1 = \frac{3}{4}, \ C_2 = -\frac{1}{4}$$

Our solution to the initial-value problem is then

$$\vec{Y}(t) = \frac{3}{4} \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \frac{1}{4} e^{-8t} \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

c. This is a special case because of the zero eigenvalue, which changes the phase portrait. All points along the line determined by the eigenvector $\vec{v}_1 = \langle 2, 1 \rangle$ are equilibrium solutions, which can be easily checked. All solutions are lines with direction $\vec{v}_2 = \langle 2, 3 \rangle$ and go towards an equilibrium solution mentioned earlier (since $\lambda_2 = -8 < 0$).



2.8 Second-Order Linear Equations

Find the general solution of the second-order equation and then find the solution to the initial-value problem.

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} - 5y = 0, \ y(0) = 11, \ y'(0) = -7$$

Solution

We will convert the second-order equation into a system of equations by letting

$$v = \frac{dy}{dt}.$$

Then we have that

$$\frac{dv}{dt} = \frac{d^2y}{dt^2} = 5y - 4\frac{dy}{dt} = 5y - 4v.$$

This gives us the following system of equations.

$$\begin{aligned} \frac{dy}{dt} &= 0y + 1v \\ \frac{dv}{dt} &= 5y - 4v \end{aligned}$$

We compute the eigenvalues of the system as normal.

$$\det \begin{pmatrix} 0-\lambda & 1 \\ 5 & -4-\lambda \end{pmatrix} = \lambda^2 + 4\lambda - 5 = (\lambda - 1)(\lambda + 5) = 0 \implies \lambda_1 = 1, \ \lambda_2 = -5$$

We compute the eigenvectors as follows.

$$\begin{aligned} \lambda_1 &: \begin{pmatrix} 0-\lambda_1 & 1 \\ 5 & -4-\lambda_1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 5 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies x = y, \ \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\ \lambda_2 &: \begin{pmatrix} 0-\lambda_2 & 1 \\ 5 & -4-\lambda_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies -5x = y, \ \vec{v}_2 = \begin{pmatrix} 1 \\ -5 \end{pmatrix} \end{aligned}$$

Remember that any multiple of these vectors will also be an eigenvector. We now write the general solution.

$$\begin{pmatrix} y(t) \\ v(t) \end{pmatrix} = C_1 e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-5t} \begin{pmatrix} 1 \\ -5 \end{pmatrix}$$

To find our solution to the initial-value problem, we plug in $t = 0$ and our initial conditions, $y(0) = 11$, $v(0) = y'(0) = -7$.

$$\begin{pmatrix} 11 \\ -7 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ -5 \end{pmatrix} \implies C_1 = 8, \ C_2 = 3$$

Our solution to the system is then

$$\begin{pmatrix} y(t) \\ v(t) \end{pmatrix} = 8e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} + 3e^{-5t} \begin{pmatrix} 1 \\ -5 \end{pmatrix} = \begin{pmatrix} 8e^t + 3e^{-5t} \\ 8e^t - 15e^{-5t} \end{pmatrix}$$

From this, we can read across the top row to get our answer to the initial-value problem.

$$y(t) = 8e^t + 3e^{-5t}$$

2.9 Trace-Determinant Plane

Consider the one-parameter family of systems of linear differential equations.

$$\frac{d\vec{Y}}{dt} = \begin{pmatrix} \alpha & 1 \\ \alpha & \alpha \end{pmatrix} \vec{Y}$$

a. Sketch the corresponding curve in the trace-determinant plane.

b. Identify bifurcation values.

c. Describe the different types of behaviors exhibited by the system as α increases along the real line.

Solution

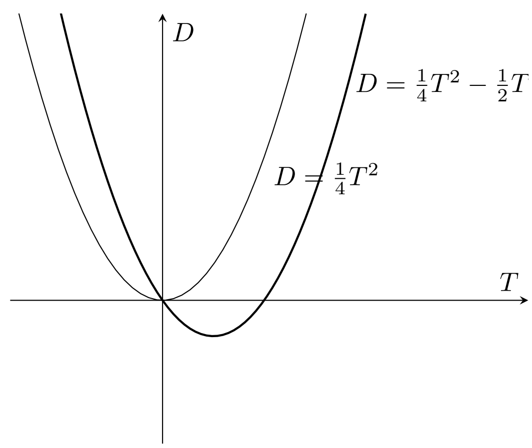
a. Before we can sketch the curve in the trace-determinant plane, we first need to compute the trace and the determinant. In our case, we have

$$T = 2\alpha, \; D = \alpha^2 - \alpha.$$

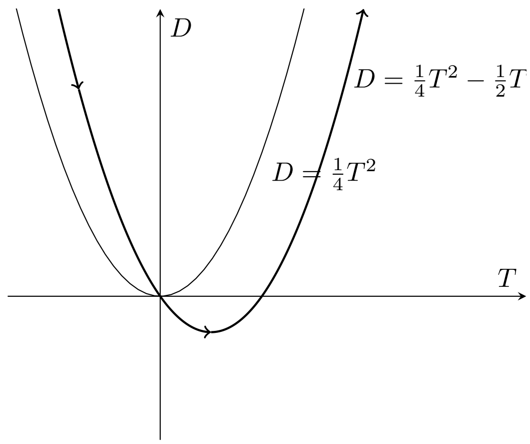
At this point, we have parametric equations for T and D. We sketch them in the trace-determinant plane like one would sketch the curve $x(t) = 2t$, $y(t) = t^2 - t$. We first eliminate the parameter α and determine the direction that the curve is traced out. To eliminate the parameter, we notice from the first equation, $\alpha = T/2$. Plugging this into the second equation, we get

$$D = \frac{1}{4}T^2 - \frac{1}{2}T.$$

We now sketch this in the trace-determinant plane.



Now all that we need is to determine the direction that the curve traverses as α increases. For $\alpha = 0$, $T = D = 0$, so we are at the origin, for $\alpha = 1$, $T = 2$ and $D = 0$. This implies as α increases, we move to the right. We include arrows in our sketch.



b. Our bifurcation values are where our sketched curve passes through the x-axis, the positive y-axis, or the parabola $D = T^2/4$. A look at our sketch shows that there are 2 bifurcation values for us to locate. The first of which is at the origin, $D = T = 0$. Solving $T = 2\alpha = 0$ shows that this corresponds to $\alpha = 0$. The second bifurcation value is where $D = 0$ and $T = 2$ (looking at the roots of our parabola). Solving $T = 2\alpha = 2$ shows us that the second bifurcation is at $\alpha = 1$.

c. For $\alpha < 0$, we are inside the left portion of the parabola $D = T^2/4$. This tells us that our solutions here will be spiral sinks here. For $\alpha = 0$, we have both the zero eigenvalue and repeated eigenvalue case (by plugging in $\alpha = 0$ into our original equation). For $0 < \alpha < 1$, we are below the y-axis. This tells us that our solutions here will be saddles. For $\alpha = 1$, we have the zero eigenvalue case. For $\alpha > 1$, we are outside the right portion of the parabola $D = T^2/4$. This tells us that our solutions here will be sources.

2.10 Forced Harmonic Oscillators

Find the general solution for each of the given equations.

a. $\dfrac{d^2y}{dt^2} + 6\dfrac{dy}{dt} + 8y = 2e^{-3t}$

b. $\dfrac{d^2y}{dt^2} + 7\dfrac{dy}{dt} + 10y = e^{-2t}$

Solution

a. We will use the method of undetermined coefficients. First, we solve for the homogeneous solution by the characteristic equation.

$$r^2 + 6r + 8 = 0 \implies r = -2, -4$$

Therefore, the homogeneous solution is

$$y_h(t) = C_1e^{-2t} + C_2e^{-4t}.$$

For our particular solution, we will make a guess of the form $y_p(t) = Ae^{-3t}$. Plugging this into our equation, we arrive at the following.

$$\begin{aligned} \left(9Ae^{-3t}\right) + 6\left(-3Ae^{-3t}\right) + 8(Ae^{-3t}) &= 2e^{-3t} \\ -Ae^{-3t} &= 2e^{-3t} \\ -A &= 2 \\ A &= -2 \end{aligned}$$

By the linearity principle, the general solution is as follows.

$$y_g(t) = y_p(t) + y_h(t) = -2e^{-3t} + C_1e^{-2t} + C_2e^{-4t}$$

b. As before, we use the method of undetermined coefficients. We first solve for the homogeneous solution by the characteristic equation.

$$r^2 + 7r + 10 = 0 \implies r = -2, -5$$

Therefore, the homogeneous solution is

$$y_h(t) = C_1e^{-2t} + C_2e^{-5t}.$$

For our particular solution, we should guess $y_p(t) = Ae^{-2t}$, however, we know this to be part of our particular solution. Because of this, we modify our guess to be $y_p(t) = Ate^{-2t}$. Plugging this into our equation, we arrive at the following.

$$\begin{aligned} \left(4Ate^{-2t} - 4Ae^{-2t}\right) + 7\left(-2Ate^{-2t} + Ae^{-2t}\right) + 10\left(Ate^{-2t}\right) &= e^{-2t} \\ 3Ae^{-2t} &= e^{-2t} \\ 3A &= 1 \\ A &= \frac{1}{3} \end{aligned}$$

By the linearity principle, the general solution is as follows.

$$y_g(t) = y_p(t) + y_h(t) = \frac{1}{3}te^{-2t} + C_1e^{-2t} + C_2e^{-5t}$$

2.11 Sinusoidal Forcing

Using the method of complexification, find the general solution for each of the given equations.

a. $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \cos t$

b. $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \sin t$

Solution

a. We must first compute the homogeneous solution by solving the characteristic equation.

$$r^2 + 3r + 2 = (r+1)(r+2) = 0 \implies r = -1, -2$$

Therefore, our homogeneous solution is

$$y_h(t) = C_1 e^{-t} + C_2 e^{-2t}.$$

Now we complexify the equation. We do so by writing e^{it} instead of $\cos t$ (since $\cos t$ is the real part of e^{it} by Euler's formula).

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = e^{it}$$

Since we already have the homogeneous solution, we need only guess the particular solution. Using undetermined coefficients, we guess $y_p(t) = Ae^{it}$. Plugging this into our equation, we get

$$\begin{aligned}\left(-Ae^{it}\right) + 3\left(iAe^{it}\right) + 2\left(Ae^{it}\right) &= e^{it}\\ (1+3i)Ae^{it} &= e^{it}\\ (1+3i)A &= e^{it}\\ A &= \frac{1}{1+3i}\\ A &= \frac{1-3i}{10}\end{aligned}$$

Therefore, the particular solution is given by

$$y_p(t) = \frac{1-3i}{10}e^{it} = \frac{1}{10}(1-3i)(\cos t + i\sin t) = \frac{1}{10}(\cos t + 3\sin t) + \frac{i}{10}(\sin t - 3\cos t)$$

Note that in the beginning, we started with the real part of the now complexified equation. That means that the real part of this solution will give us the solution that we are looking for. Our general solution to the original equation is then the sum of the real part of the solution above, plus the homogeneous solution.

$$y_g(t) = \text{Re}(y_p(t)) + y_h(t) = \frac{1}{10}\cos t + \frac{3}{10}\sin t + C_1 e^{-t} + C_2 e^{-2t}$$

b. Now that we have the same equation as the previous problem but with $\sin t$. Since $\sin t$ is the imaginary part of e^{it}, we apply the same reasoning as above. In this case, however, since we started with the imaginary part of the complexified equation, we will take the imaginary part of the particular solution (the particular solution from the previous problem). Since the homogeneous solution stays the same, we have the general solution.

$$y_g(t) = y_p(t) + y_h(t) = \sin t - 3\cos t + C_1 e^{-t} + C_2 e^{-2t}$$

2.12 Undamped Forcing and Resonance

Consider the equation below.

$$\frac{d^2y}{dt^2} + 6y = \cos 2t$$

a. Compute the general solution of the differential equation.

b. Determine if the observed phenomena corresponds to beating or resonance.

c. Find the frequency and period of the rapid oscillations and beating (if applicable).

d. Draw a rough sketch of a general solution curve.

Solution

a. We first find the homogeneous solution by solving the characteristic equation.

$$r^2 + 6 = 0 \implies r = \pm i\sqrt{6}$$

Using Euler's formula, we have

$$e^{i\sqrt{6}t} = \cos\sqrt{6}t + i\sin\sqrt{6}t.$$

Both the real and imaginary parts of this are solutions and they are independent. Therefore, the homogeneous solution can be written as

$$y_h(t) = C_1\cos\sqrt{6}t + C_2\sin\sqrt{6}t$$

To determine the particular solution, either undetermined coefficients or complexification can be used. Either method gives the result of

$$y_p(t) = \frac{1}{2}\cos 2t.$$

Therefore, the general solution is given by

$$y_g(t) = y_p(t) + y_h(t) = \frac{1}{2}\cos 2t + C_1\cos\sqrt{6}t + C_2\sin\sqrt{6}t$$

b. Our phenomena corresponds to beating, as can be seen from two places. Since there are two frequencies in our solution, they can be combined to a beating term. Similarly, if resonance were to occur, there would be a term of the form $At\sin\sqrt{6}t$ or $At\cos\sqrt{6}t$.

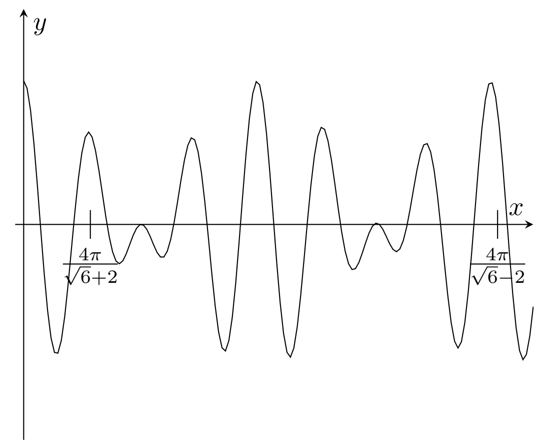
c. The frequency and period of the rapid oscillations are given by

$$\text{Frequency : } \left|\frac{\sqrt{6}+2}{4\pi}\right| \qquad \text{Period : } \left|\frac{4\pi}{\sqrt{6}+2}\right|$$

Similarly, the frequency and period of the beats are given by

$$\text{Frequency : } \left|\frac{\sqrt{6}-2}{4\pi}\right| \qquad \text{Period : } \left|\frac{4\pi}{\sqrt{6}-2}\right|$$

d. A general solution curve looks similar to the following.



y
x
$\frac{4\pi}{\sqrt{6}+2}$
$\frac{4\pi}{\sqrt{6}-2}$

Consider the equation below.

$$\frac{d^2y}{dt^2} + 9y = \cos 3t$$

a. Compute the general solution of the differential equation.

b. Determine if the observed phenomena corresponds to beating or resonance.

c. Find the frequency and period of the rapid oscillations and beating (if applicable).

d. Draw a rough sketch of a general solution curve.

Solution

a. As the previous problem, we compute the homogeneous solution via the characteristic equation.

$$r^2 + 9 = 0 \implies r = \pm 3i \implies y_h(t) = C_1 \cos 3t + C_2 \sin 3t$$

To solve for the particular solution, we undetermined coefficients can be used, with a modified guess.

$$y_p(t) = \frac{1}{6} t \sin 3t$$

The general solution is then the following.

$$y_g(t) = y_p(t) + y_h(t) = \frac{1}{6} t \sin 3t + C_1 \cos 3t + C_2 \sin 3t$$

b. This time, our phenomena corresponds to resonance, which again can be seen in two ways. First, the particular solution has the same frequency as the homogeneous solution. Second, we have a term of the form $At \sin 3t$ or $Bt \cos 3t$ in our solution.

c. The frequency and period of the rapid oscillations are given by

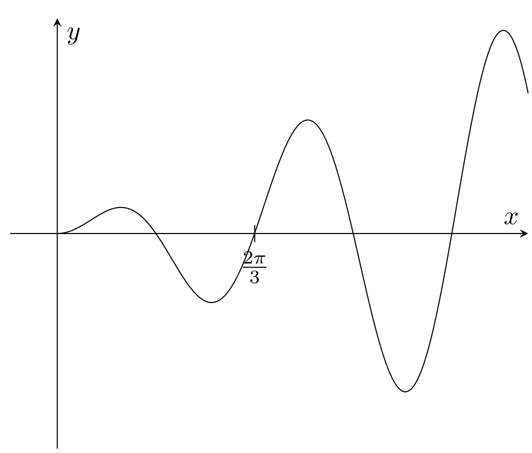
$$\text{Frequency}: \left|\frac{3+3}{4\pi}\right| = \frac{3}{2\pi} \qquad \text{Period}: \left|\frac{4\pi}{3+3}\right| = \frac{2\pi}{3}$$

Similarly, the frequency of the beats is given by

$$\text{Frequency}: \left|\frac{3-3}{4\pi}\right| = 0$$

Note that there is no period of the beating!

d. A general solution curve looks similar to the following.



2.13 Amplitude and Phase of Solutions

Compute the general solution to the equation below in amplitude-phase form.

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 2\cos 3t$$

What is the steady-state solution to this problem?

Solution

To compute the solution in amplitude-phase notation, we first compute the general solution as normal. To do so, we solve the characteristic equation and arrive at $r = -1, -2$. The homogeneous solution is then

$$y_h(t) = C_1e^{-t} + C_2e^{-2t}.$$

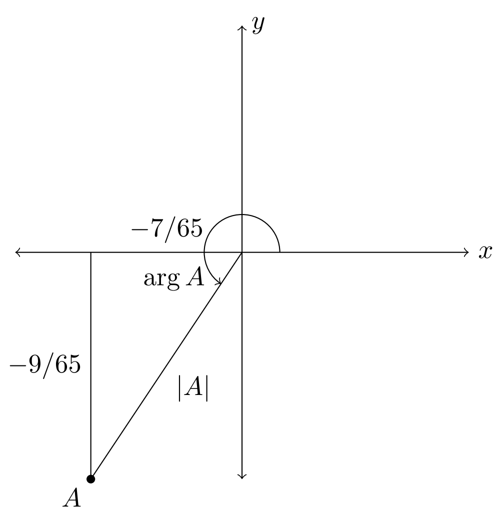
To solve for the particular solution, we now use complexification. In other words, we look for a particular solution of

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 2e^{3it}.$$

We use undetermined coefficients and make out guess $y_p(t) = Ae^{3it}$. Plugging this into our equation, we arrive at

$$\begin{aligned}\left(-9Ae^{3it}\right) + 3\left(3iAe^{3it}\right) + 2\left(Ae^{3it}\right) &= 2e^{3it}\\ (-7+9i)Ae^{3it} &= 2e^{3it}\\ (-7+9i)A &= 2\\ A &= \frac{2}{-7+9i}\\ A &= \frac{-7-9i}{65}\end{aligned}$$

Now, we write the complex number A in polar form. That is, we write $A = re^{i\theta}$ where $r = |A|$ and $\tan\theta = \arg A$. This is most easily done by drawing the point A in the complex plane ($z = x + iy$).



From this picture, we see that

$$r = \sqrt{\left(\frac{-7}{65}\right)^2 + \left(\frac{-9}{65}\right)^2} = \frac{\sqrt{130}}{65},\ \theta = \tan^{-1}\left(\frac{-9/65}{-7/65}\right) + \pi = \tan^{-1}\frac{9}{7} + \pi.$$

So we have

$$A = \frac{\sqrt{130}}{65} e^{i\theta}.$$

Then the particular solution to the complexified equation can be written as

$$y_p(t) = \frac{\sqrt{130}}{65} e^{i\theta} e^{3it} = \frac{\sqrt{130}}{65} e^{i(3t+\theta)}.$$

Since our equation was initially the real part of the complexified equation, we take the real part of the solution to get our particular solution to the original problem. Our general solution can then be written as

$$y_g(t) = \frac{\sqrt{130}}{65} \cos(3t + \theta) + C_1 e^{-t} + C_2 e^{-2t}, \ \theta = \tan^{-1} \frac{9}{7} + \pi$$

3 Laplace Transformations

3.1 Inverse Laplace Transformations

Compute the following inverse Laplace transformations.

a. $\mathcal{L}^{-1}\left(\frac{3}{2s-1}\right)$

b. $\mathcal{L}^{-1}\left(\frac{s+4}{s^2+4}\right)$

c. $\mathcal{L}^{-1}\left(\frac{s}{(s+1)(s^2+1)}\right)$

Solution

a. From the table of Laplace transforms, we have the following.

$$\mathcal{L}^{-1}\left(\frac{3}{2s-1}\right) = \frac{3}{2}\mathcal{L}^{-1}\left(\frac{1}{s-\frac{1}{2}}\right) = \frac{3}{2}e^{t/2}$$

b. We need to split up the problem into two parts.

$$\mathcal{L}^{-1}\left(\frac{s+4}{s^2+4}\right) = \mathcal{L}^{-1}\left(\frac{s}{s^2+4}\right) + \mathcal{L}^{-1}\left(\frac{4}{s^2+4}\right)$$

Now use the transformation formulas for sine and cosine.

$$\mathcal{L}^{-1}\left(\frac{s}{s^2+4}\right) + \mathcal{L}^{-1}\left(\frac{4}{s^2+4}\right) = \mathcal{L}^{-1}\left(\frac{s}{s^2+2^2}\right) + 2\mathcal{L}^{-1}\left(\frac{2}{s^2+2^2}\right) = \cos 2t + 2\sin 2t$$

c. First, use partial fractions.

$$\frac{s}{(s+1)(s^2+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+1}$$

Multiplying through by $(s+1)(s^2+1)$, we get a system of equations and we get that $A = -1/2$, $B = 1/2$, and $C = 1/2$. Therefore, we have that

$$\frac{s}{(s+1)(s^2+1)} = \frac{-1/2}{s+1} + \frac{s/2+1/2}{s^2+1}.$$

We then split the inverse Laplace of this into three different parts.

$$\begin{aligned}\mathcal{L}^{-1}\left(\frac{s}{(s+1)(s^2+1)}\right) &= \mathcal{L}^{-1}\left(\frac{-1/2}{s+1}\right) + \mathcal{L}^{-1}\left(\frac{s/2}{s^2+1}\right) + \mathcal{L}^{-1}\left(\frac{1/2}{s^2+1}\right)\\ &= \frac{-1}{2}\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) + \frac{1}{2}\mathcal{L}^{-1}\left(\frac{s}{s^2+1^2}\right) + \frac{1}{2}\mathcal{L}^{-1}\left(\frac{1}{s^2+1^2}\right)\\ &= -\frac{1}{2}e^{-t} + \frac{1}{2}\cos t + \frac{1}{2}\sin t\end{aligned}$$

3.2 Solutions to Linear Equations

Use the method of Laplace transformations to compute the solution to the given initial-value problem.

$$\frac{dy}{dt} + 4y = 2 + 4t,\ y(0) = 1$$

Solution

Apply the Laplace transformation to both sides of the equation and solve for $\mathcal{L}(y(t))$.

$$\begin{aligned}
\mathcal{L}\left(\frac{dy}{dt} + 4y\right) &= \mathcal{L}(2 + 4t) \\
\mathcal{L}\left(\frac{dy}{dt}\right) + \mathcal{L}(4y) &= \mathcal{L}(2) + \mathcal{L}(4t) \\
\mathcal{L}\left(\frac{dy}{dt}\right) + 4\mathcal{L}(y) &= 2\mathcal{L}(1) + 4\mathcal{L}(t) \\
s\mathcal{L}(y) - y(0) + 4\mathcal{L}(y) &= \frac{2}{s} + \frac{4}{s^2} \\
(s+4)\mathcal{L}(y) &= \frac{2}{s} + \frac{4}{s^2} + 1 \\
\mathcal{L}(y) &= \frac{2}{s(s+4)} + \frac{4}{s^2(s+4)} + \frac{1}{s+4}
\end{aligned}$$

Now we need to apply partial fractions to be able to apply the inverse Laplace transformation.

$$\frac{2}{s(s+4)} = \frac{1/2}{s} - \frac{1/2}{s+4},\ \frac{4}{s^2(s+4)} = \frac{-1/4}{s} + \frac{1}{s^2} + \frac{1/4}{s+4}$$

Then we have the following.

$$\mathcal{L}(y) = \frac{1/4}{s} + \frac{1}{s^2} + \frac{3/4}{s+4}$$

Now apply the inverse Laplace transformation to both sides of the equation to solve for $y(t)$.

$$\begin{aligned}
y(t) &= \frac{1}{4}\mathcal{L}^{-1}\left(\frac{1}{s}\right) + \mathcal{L}^{-1}\left(\frac{1}{s^2}\right) + \frac{1}{4}\mathcal{L}^{-1}\left(\frac{1}{s+4}\right) \\
&= \frac{1}{4} + t + \frac{1}{4}e^{-4t}
\end{aligned}$$

Use the method of Laplace transformations to compute the solution to the given initial-value problem.

$$\frac{d^2y}{dt^2} + 4y = \cos 5t,\ y(0) = 0,\ y'(0) = -2$$

Solution

As before, apply the Laplace transformation to both sides and solve for $\mathcal{L}(y(t))$.

$$\begin{aligned}
\mathcal{L}\left(\frac{d^2y}{dt^2} + 4y\right) &= \mathcal{L}(\cos 5t) \\
\mathcal{L}\left(\frac{d^2y}{dt^2}\right) + \mathcal{L}(4y) &= \mathcal{L}(\cos 5t) \\
\mathcal{L}\left(\frac{d^2y}{dt^2}\right) + 4\mathcal{L}(y) &= \mathcal{L}(\cos 5t) \\
s^2\mathcal{L}(y) - sy(0) - y'(0) + 4\mathcal{L}(y) &= \frac{s}{s^2+25} \\
(s^2+4)\mathcal{L}(y) &= -2 + \frac{s}{s^2+25} \\
\mathcal{L}(y) &= \frac{-2}{s^2+4} + \frac{s}{(s^2+4)(s^2+25)}
\end{aligned}$$

Now we apply partial fractions on the right to get

$$\frac{s}{(s^2+4)(s^2+25)} = \frac{s/21}{s^2+4} + \frac{-s/21}{s^2+25}.$$

Plugging this in, we use our table to solve for $y(t)$.

$$\begin{aligned}
\mathcal{L}(y) &= \frac{-2}{s^2+4} + \frac{s/21}{s^2+4} - \frac{s/21}{s^2+25} \\
y(t) &= -\mathcal{L}^{-1}\left(\frac{2}{s^2+2^2}\right) + \frac{1}{21}\mathcal{L}^{-1}\left(\frac{s}{s^2+4}\right) - \frac{1}{21}\mathcal{L}^{-1}\left(\frac{s}{s^2+25}\right) \\
y(t) &= -\sin 2t + \frac{1}{21}\cos 2t - \frac{1}{21}\cos 5t
\end{aligned}$$

3.3 Discontinuous Functions

Solve the initial-value problem.

$$\frac{dy}{dt} + y = u_2(t)e^{-2(t-2)}, \; y(0) = 0$$

Solution

We take the Laplace of both sides and solve for $\mathcal{L}(y)$, as before.

$$\begin{aligned}
\mathcal{L}\left(\frac{dy}{dt} + y\right) &= \mathcal{L}(u_2(t)e^{-2(t-2)}) \\
\mathcal{L}\left(\frac{dy}{dt}\right) + \mathcal{L}(y) &= \mathcal{L}(u_2(t)e^{-2(t-2)}) \\
s\mathcal{L}(y) - y(0) + \mathcal{L}(y) &= \mathcal{L}(u_2(t)e^{-2(t-2)}) \\
(s+1)\mathcal{L}(y) &= \frac{e^{-2s}}{s+2} \\
\mathcal{L}(y) &= \frac{e^{-2s}}{(s+1)(s+2)}
\end{aligned}$$

Note that we used the formula $\mathcal{L}(u_c(t)f(t-c)) = e^{-cs}\mathcal{L}(f(t))$ along the way. Now, we will use partial fractions, ignoring the e^{-2s} term initially.

$$\frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$$

This turns our equation into the following.

$$\mathcal{L}(y) = e^{-2s}\frac{1}{s+1} - e^{-2s}\frac{1}{s+2}$$

Now, we use the formula from before to take the inverse Laplace

$$\mathcal{L}(u_c(t)f(t-c)) = e^{-cs}\mathcal{L}(f(t))$$

We see that for the first term, $c = 2$ and $f(t) = e^{-t}$ (since $\mathcal{L}(f(t)) = 1/(s+1)$). Similarly, in the second term, $c = 2$ and $f(t) = e^{-2t}$. Our solution to the initial-value problem is then

$$y(t) = u_2(t)e^{-(t-2)} - u_2(t)e^{-2(t-2)}$$

Solve the initial-value problem.

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = \delta_3(t),\ y(0) = 1,\ y'(0) = 1$$

Solution

Start by taking the Laplace of both sides and solving for $\mathcal{L}(y(t))$.

$$\begin{aligned}
\mathcal{L}\left(\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y\right) &= \mathcal{L}(\delta_3(t)) \\
\mathcal{L}\left(\frac{d^2y}{dt^2}\right) + \mathcal{L}\left(2\frac{dy}{dt}\right) + \mathcal{L}(y) &= \mathcal{L}(\delta_3(t)) \\
\mathcal{L}\left(\frac{d^2y}{dt^2}\right) + 2\mathcal{L}\left(\frac{dy}{dt}\right) + \mathcal{L}(y) &= \mathcal{L}(\delta_3(t)) \\
s^2\mathcal{L}(y) - sy(0) - y'(0) + 2s\mathcal{L}(y) - 2y(0) + \mathcal{L}(y) &= \mathcal{L}(\delta_3(t)) \\
(s^2 + 2s + 1)\mathcal{L}(y) &= s + 3 + e^{-3s} \\
\mathcal{L}(y) &= \frac{s+3}{s^2+2s+1} + \frac{e^{-3s}}{s^2+2s+1} \\
\mathcal{L}(y) &= \frac{s+3}{(s+1)^2} + \frac{e^{-3s}}{(s+1)^2}
\end{aligned}$$

We don't have the first term in terms of common Laplace transforms, so we use partial fractions.

$$\frac{s+3}{(s+1)^2} = \frac{1}{s+1} + \frac{2}{(s+1)^2}$$

Now we have

$$\mathcal{L}(y) = \frac{1}{s+1} + \frac{2}{(s+1)^2} + \frac{e^{-3s}}{(s+1)^2}.$$

The first term is common, $\mathcal{L}(e^{-t}) = 1/(s+1)$. The second term is a little less common, but is of the form

$$\mathcal{L}(t^n e^{at}) = \frac{n!}{(s-a)^{n+1}}.$$

Therefore, the second term is $2\mathcal{L}(te^{-t})$. For the final term, we use the formula

$$\mathcal{L}(u_c(t)f(t-c)) = e^{-cs}\mathcal{L}(f(t)).$$

In our case, $c = 3$ and $f(t) = te^{-t}$ like before. Our solution is then

$$y(t) = e^{-t} + 2te^{-t} + u_3(t)(t-3)e^{-(t-3)}$$

3.4 Convolutions

Compute the convolution $f * g$ and show that $\mathcal{L}(f * g) = \mathcal{L}(f) \cdot \mathcal{L}(g)$.

$$f(t) = \cos t,\ g(t) = u_2(t)$$

Solution

The convolution $f * g$ is defined as

$$(f * g)(x) = \int_0^x f(x-t)g(t)\,dt = \int_0^x \cos(x-t)u_2(t)\,dt = \int_2^x \cos(x-t)\,dt = -\sin(x-t)\Big|_2^x = \sin(x-2)$$

From our Laplace table, we use the formula

$$\mathcal{L}(f(t-a)) = e^{-as}\mathcal{L}(f(t))$$

to get

$$\mathcal{L}(f * g) = \mathcal{L}(\sin(x-2)) = e^{-2s}\mathcal{L}(\sin x) = \frac{e^{-2s}}{s^2+1}.$$

Now we compute $\mathcal{L}(f)$ and $\mathcal{L}(g)$.

$$\mathcal{L}(f) = \mathcal{L}(\cos t) = \frac{s}{s^2+1},\ \mathcal{L}(g) = \mathcal{L}(u_2(t)) = \frac{e^{-2s}}{s}$$

Therefore, we have

$$\mathcal{L}(f) \cdot \mathcal{L}(g) = \frac{s}{s^2+1} \cdot \frac{e^{-2s}}{s} = \frac{e^{-2s}}{s^2+1} = \mathcal{L}(f * g)$$