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MATHEMATICS ASSISTANCE CENTER

# 1 First-Order Differential Equations

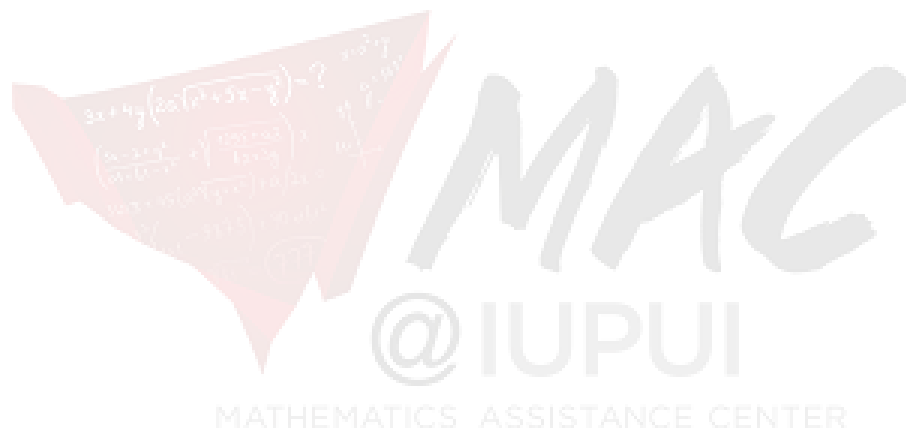
## 1.1 Separation of Variables

Find the general solution to the given differential equations

a.  $\frac{dy}{dt} = \frac{1}{2y+1}$

b.  $\frac{dy}{dt} = (y^2 + 1)t$

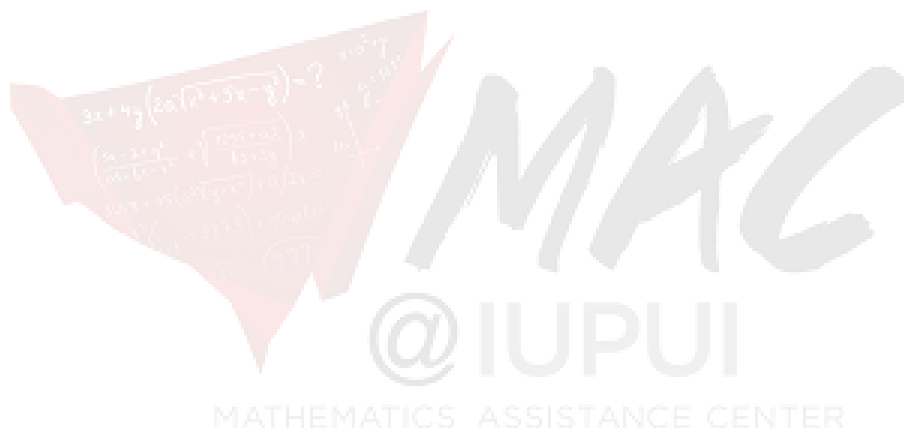
c.  $\frac{dy}{dt} = \frac{1}{ty + t + y + 1}$



## 1.2 Slope Fields

Choose points  $(t, y)$  with  $-2 \leq t \leq 2$  and  $-2 \leq y \leq 2$  and plot part of the slope field determined by the differential equation without the use of technology.

$$\frac{dy}{dt} = 4y^2$$



### 1.3 Equilibrium Solutions and Phase Lines

Given the differential equation

$$\frac{dy}{dt} = y^2 - 4y - 12,$$

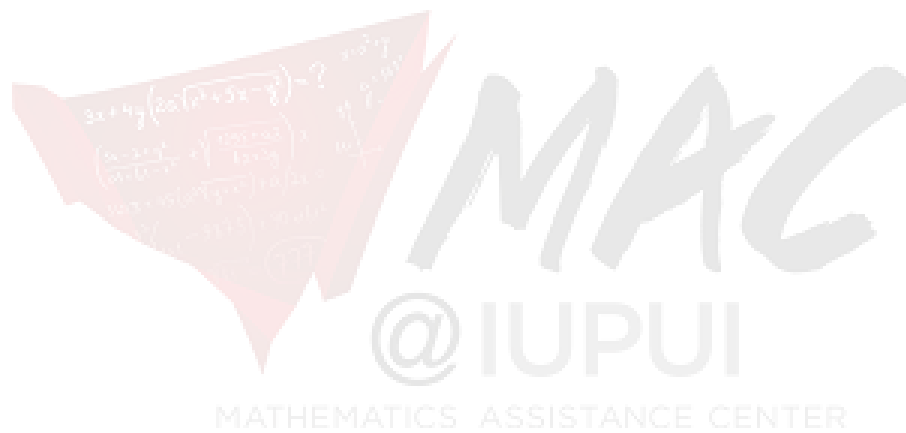
sketch the graphs of the solutions satisfying the following initial conditions.

a.  $y(0) = 1$

c.  $y(2) = 5$

b.  $y(0) = 6$

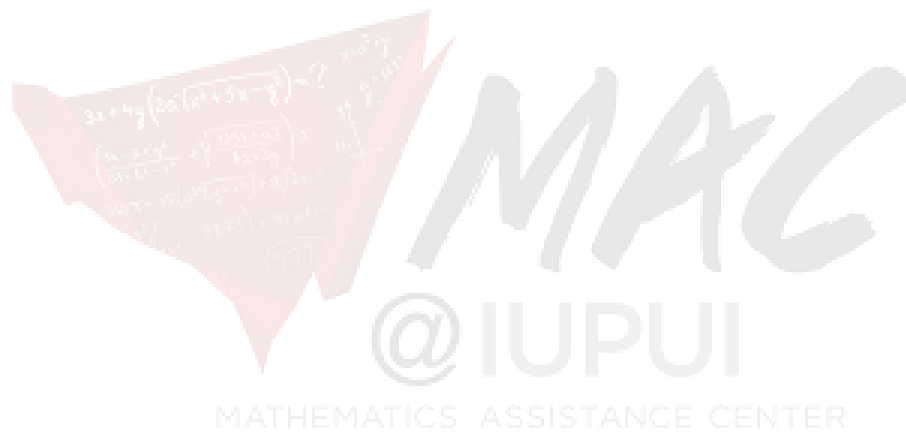
d.  $y(1) = 7$



### 1.4 Bifurcations

Locate bifurcation values for the one-parameter family of differential equations and draw phase lines to illustrate your results.

$$\frac{dy}{dt} = (y^2 - \alpha)(y^2 - 4)$$

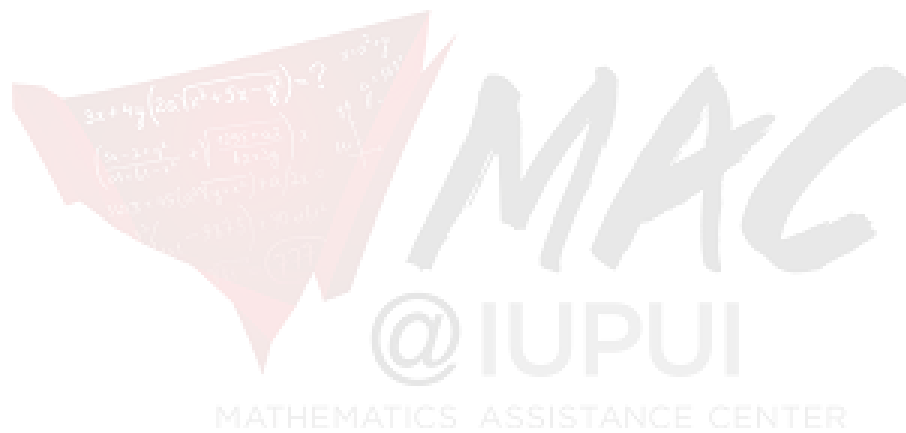


**1.5 Linear Differential Equations**

Find the general solution to the given differential equations.

a.  $\frac{dy}{dt} = -4y + 9e^{-t}$

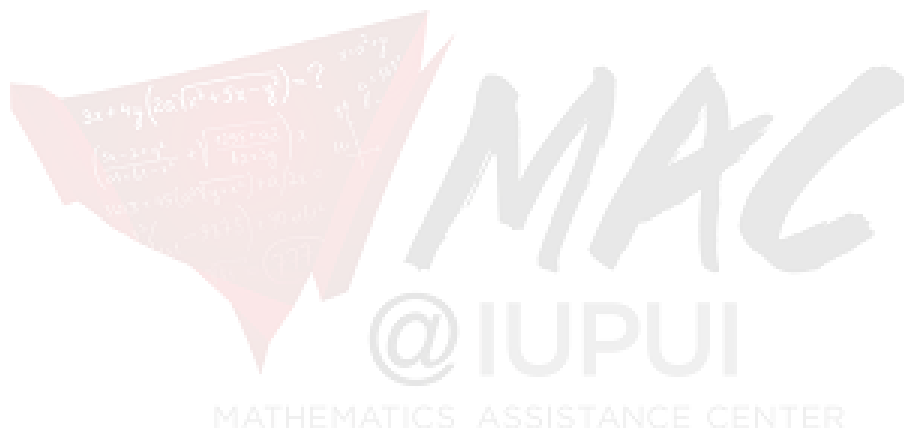
b.  $\frac{dy}{dt} = \frac{1}{2}y + 4e^{t/2}$



**1.6 Integrating Factors**

Solve the given initial-value problem.

$$t \frac{dy}{dt} - 2y = 2t^3, \quad y(-2) = 4$$

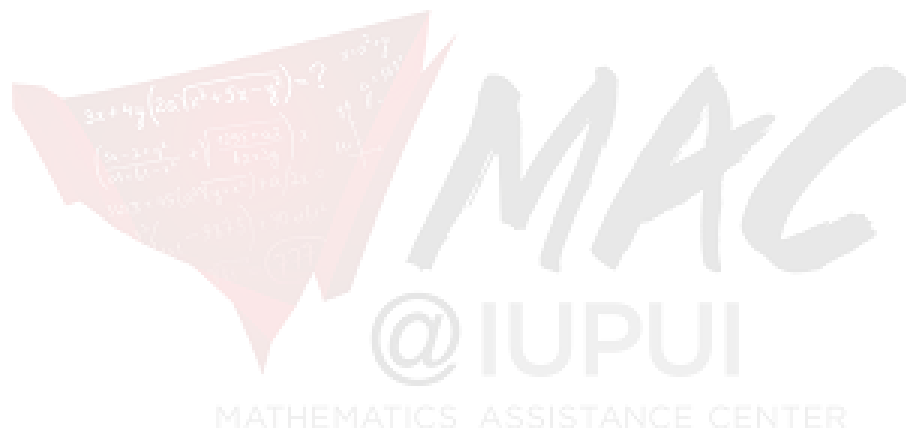


## 2 Systems of Differential Equations

### 2.1 Damped Harmonic Oscillators

Use the method of undetermined coefficients to find two non-zero solutions that are not multiples of one another.

$$\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 10y = 0$$



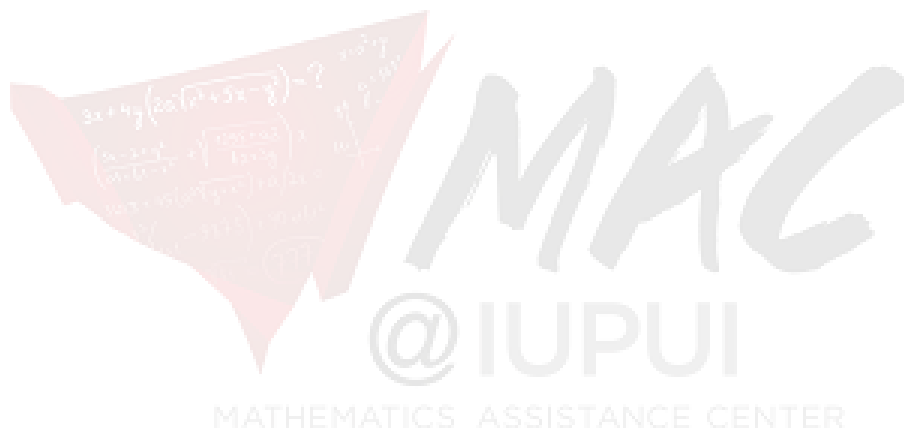


## 2.2 Decoupled Systems

Consider the following system of differential equations.

$$\begin{aligned}\frac{dx}{dt} &= 2x - 8y^2 \\ \frac{dy}{dt} &= -3y\end{aligned}$$

Derive the general solution and find the solution that corresponds to the initial condition  $(x_0, y_0) = (0, 1)$ .



### 2.3 Systems of Linear Equations

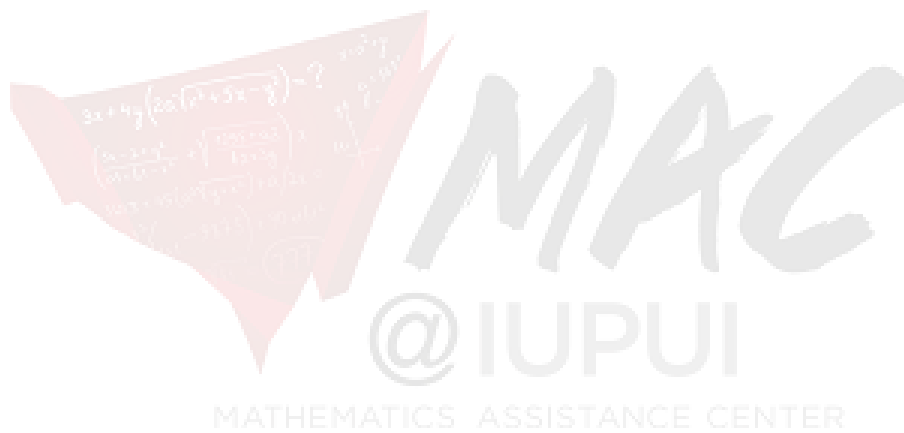
For the given system of linear equations

$$\begin{aligned}\frac{dx}{dt} &= -2x - y \\ \frac{dy}{dt} &= 2x - 5y,\end{aligned}$$

form the matrix equation that represents it and check that the functions  $\vec{Y}_1(t)$  and  $\vec{Y}_2(t)$  are solutions. If the two solutions are linearly independent, form the general solution.

$$\vec{Y}_1(t) = \langle e^{-3t} - 2e^{-4t}, e^{-3t} - 4e^{-4t} \rangle$$

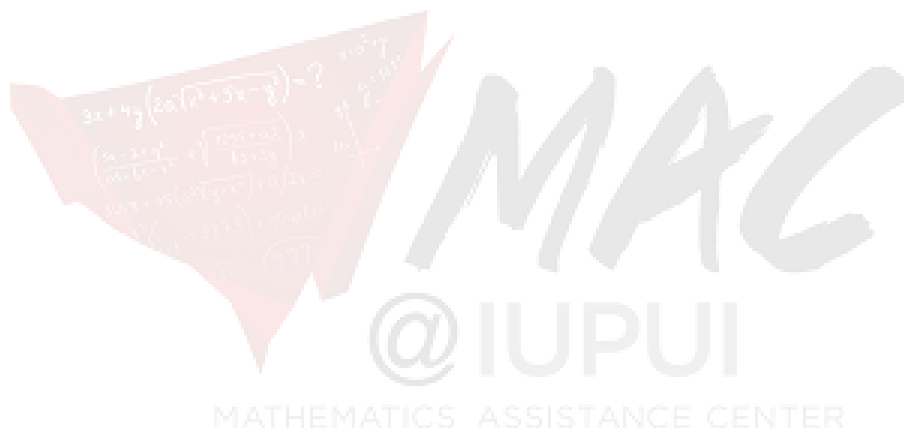
$$\vec{Y}_2(t) = \langle 2e^{-3t} + e^{-4t}, 2e^{-3t} + 2e^{-4t} \rangle$$



**2.4 Straight-Line Solutions**

Solve the initial-value problem.

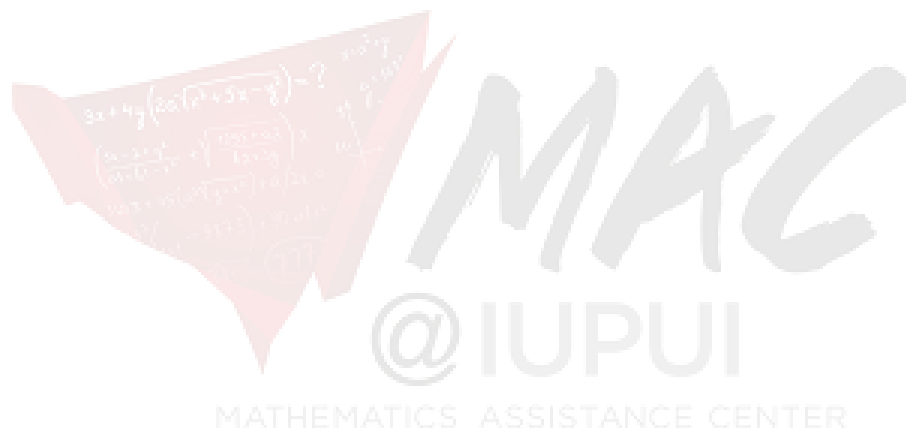
$$\frac{d\vec{Y}}{dt} = \begin{pmatrix} -4 & 1 \\ 2 & -3 \end{pmatrix} \vec{Y}, \quad \vec{Y}(0) = \langle 2, 1 \rangle$$



## 2.5 Phase Portraits

Sketch the solution curves in the phase plane and the  $x(t)$  and  $y(t)$  graphs corresponding to the initial-value problem.

$$\frac{d\vec{Y}}{dt} = \begin{pmatrix} -2 & -2 \\ -2 & 1 \end{pmatrix} \vec{Y}, \quad \vec{Y}(0) = \langle 3, -1 \rangle$$

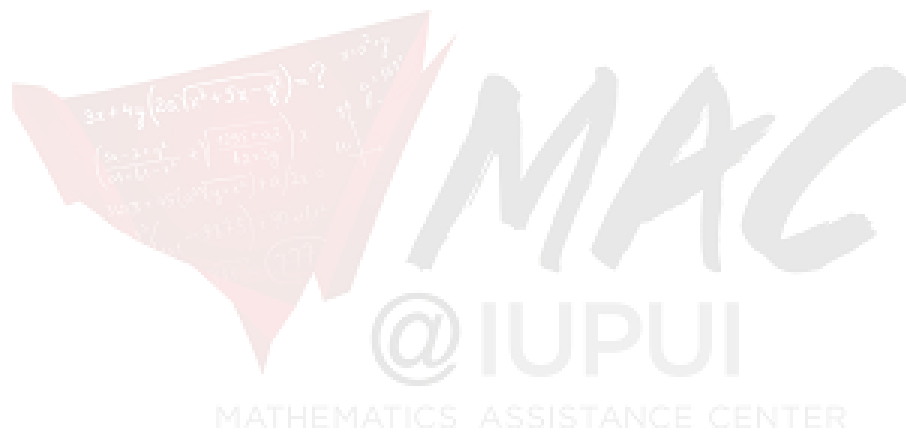


## 2.6 Complex Eigenvalues

Consider the system below.

$$\frac{d\vec{Y}}{dt} = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix} \vec{Y}, \quad \vec{Y}(0) = \langle 2, 1 \rangle$$

- Compute the eigenvalues.
- Determine if the origin is a spiral sink, spiral source, or center.
- Determine the natural period and the natural frequency.
- Determine the direction of the oscillations in the phase plane.
- Write the general solution and a solution to the initial-value problem.
- Draw the solution in the phase plane.

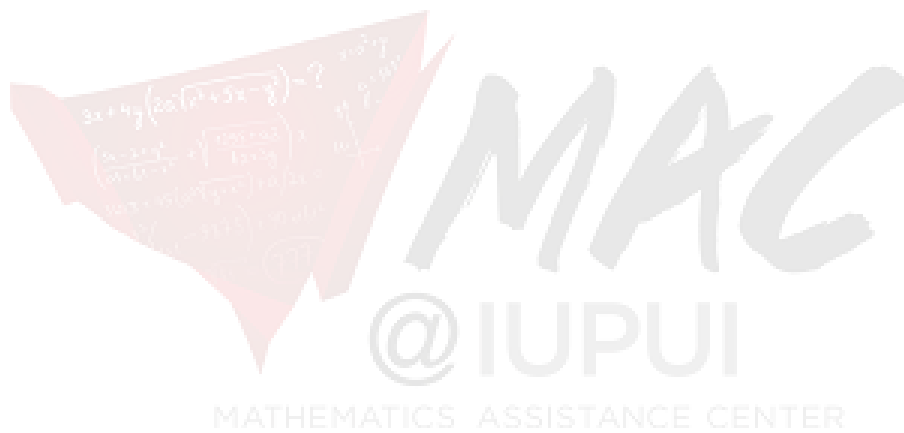


## 2.7 Special Cases

Consider the system below.

$$\frac{d\vec{Y}}{dt} = \begin{pmatrix} -2 & -1 \\ 1 & -4 \end{pmatrix} \vec{Y}, \quad \vec{Y}(0) = \langle 1, 0 \rangle$$

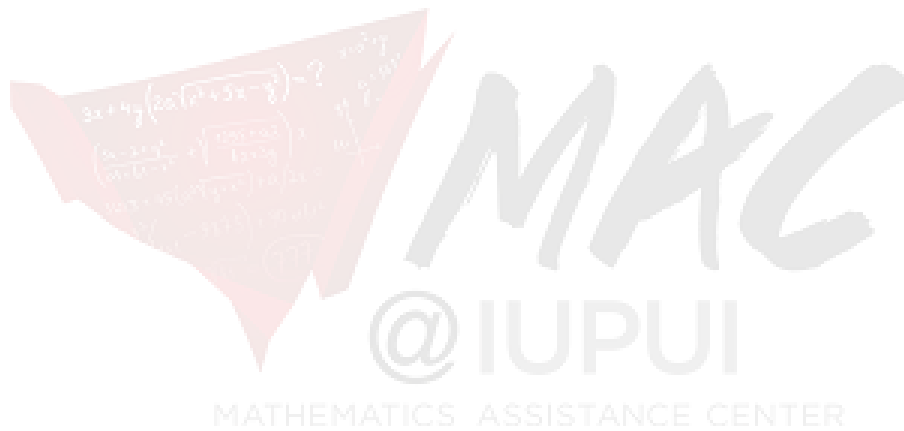
- Compute the eigenvalues and associated eigenvectors.
- Write the general solution and the solution to the initial-value problem.
- Draw solutions in the phase plane.



Consider the system below.

$$\frac{d\vec{Y}}{dt} = \begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix} \vec{Y}, \quad \vec{Y}(0) = \langle 1, 0 \rangle$$

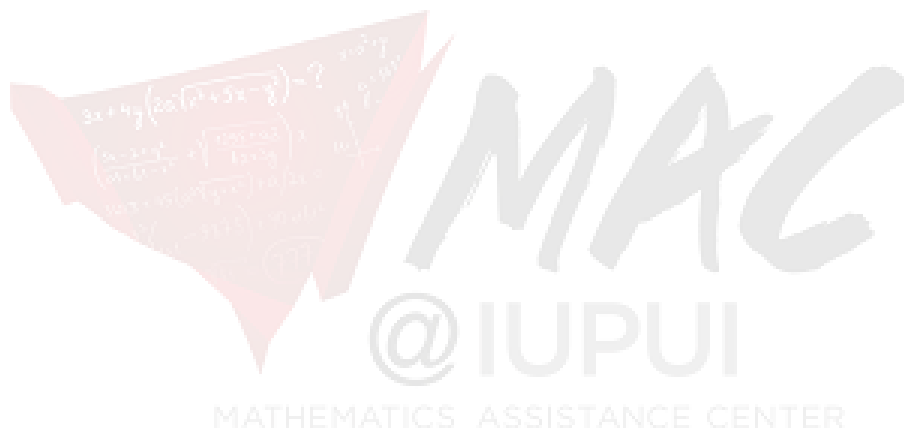
- Compute the eigenvalues and associated eigenvectors.
- Write the general solution and the solution to the initial-value problem.
- Draw solutions in the phase plane.



**2.8 Second-Order Linear Equations**

Find the general solution of the second-order equation and then find the solution to the initial-value problem.

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} - 5y = 0, \quad y(0) = 11, \quad y'(0) = -7$$



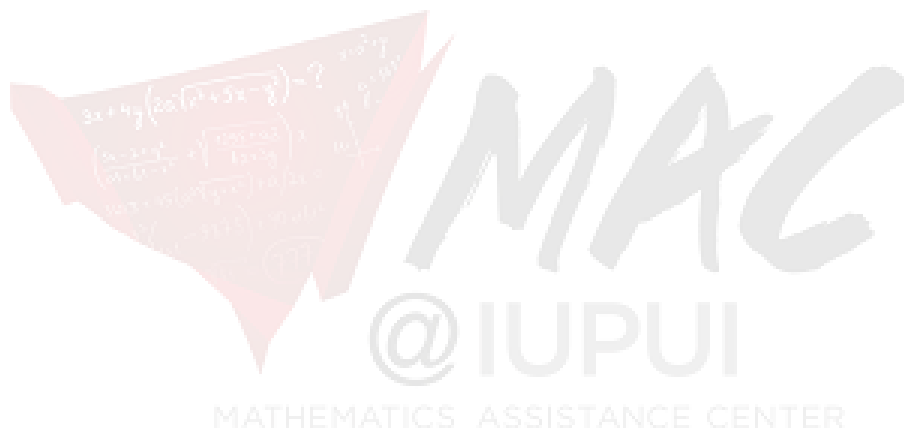


## 2.9 Trace-Determinant Plane

Consider the one-parameter family of systems of linear differential equations.

$$\frac{d\vec{Y}}{dt} = \begin{pmatrix} \alpha & 1 \\ \alpha & \alpha \end{pmatrix} \vec{Y}$$

- Sketch the corresponding curve in the trace-determinant plane.
- Identify bifurcation values.
- Describe the different types of behaviors exhibited by the system as  $\alpha$  increases along the real line.

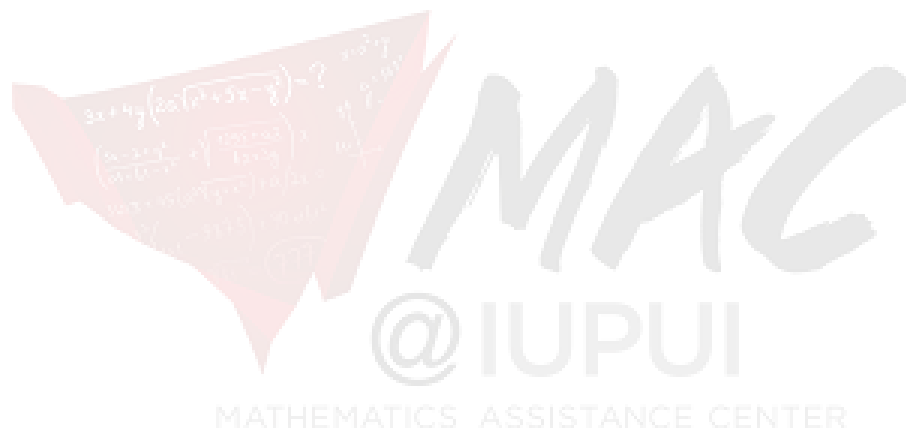


**2.10 Forced Harmonic Oscillators**

Find the general solution for each of the given equations.

a.  $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 8y = 2e^{-3t}$

b.  $\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 10y = e^{-2t}$

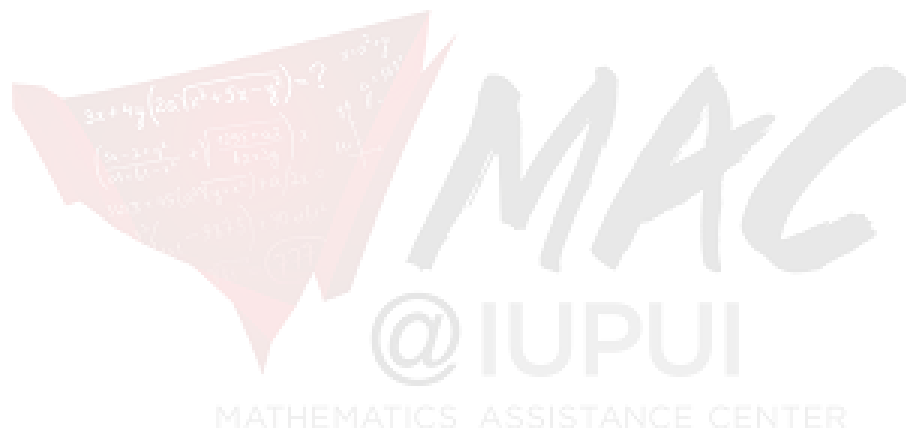


**2.11 Sinusoidal Forcing**

Using the method of complexification, find the general solution for each of the given equations.

a.  $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \cos t$

b.  $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \sin t$

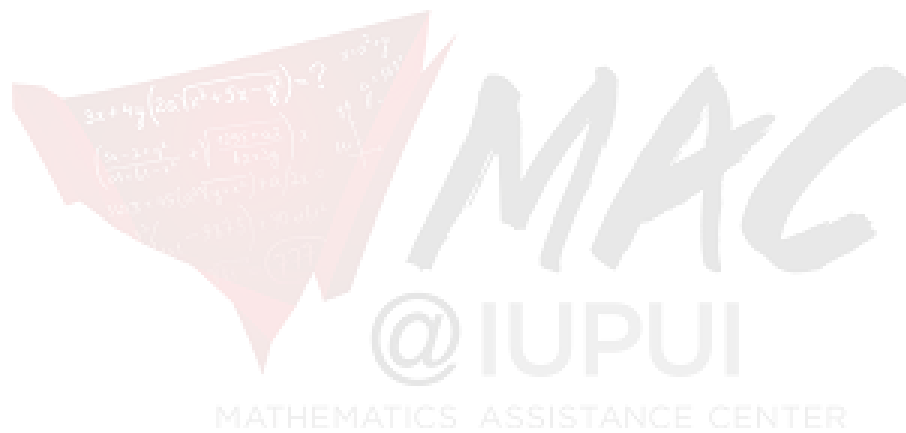


**2.12 Undamped Forcing and Resonance**

Consider the equation below.

$$\frac{d^2y}{dt^2} + 6y = \cos 2t$$

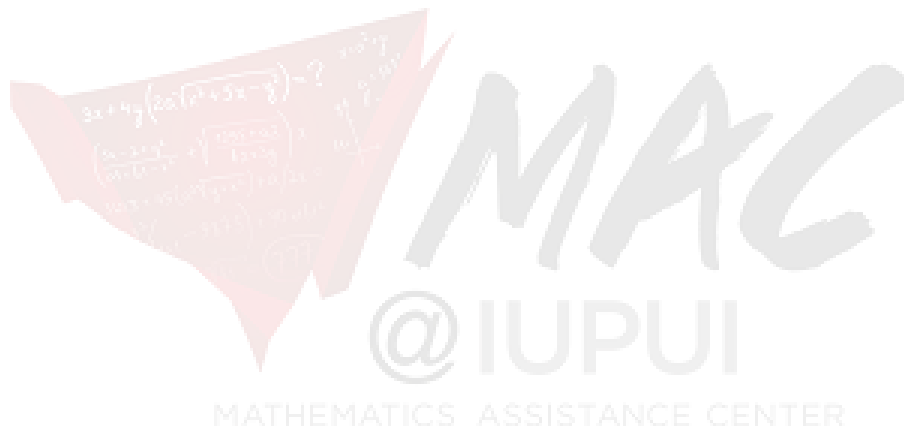
- Compute the general solution of the differential equation.
- Determine if the observed phenomena corresponds to beating or resonance.
- Find the frequency and period of the rapid oscillations and beating (if applicable).
- Draw a rough sketch of a general solution curve.



Consider the equation below.

$$\frac{d^2y}{dt^2} + 9y = \cos 3t$$

- Compute the general solution of the differential equation.
- Determine if the observed phenomena corresponds to beating or resonance.
- Find the frequency and period of the rapid oscillations and beating (if applicable).
- Draw a rough sketch of a general solution curve.

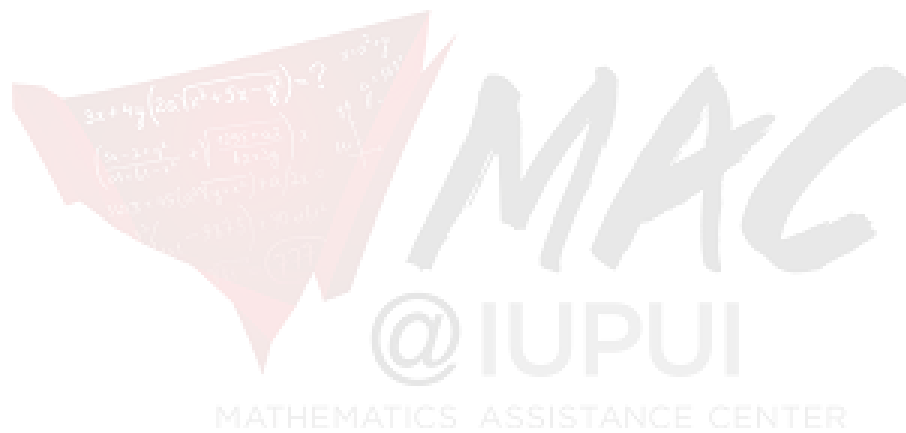


**2.13 Amplitude and Phase of Solutions**

Compute the general solution to the equation below in amplitude-phase form.

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 2 \cos 3t$$

What is the steady-state solution to this problem?



### 3 Laplace Transformations

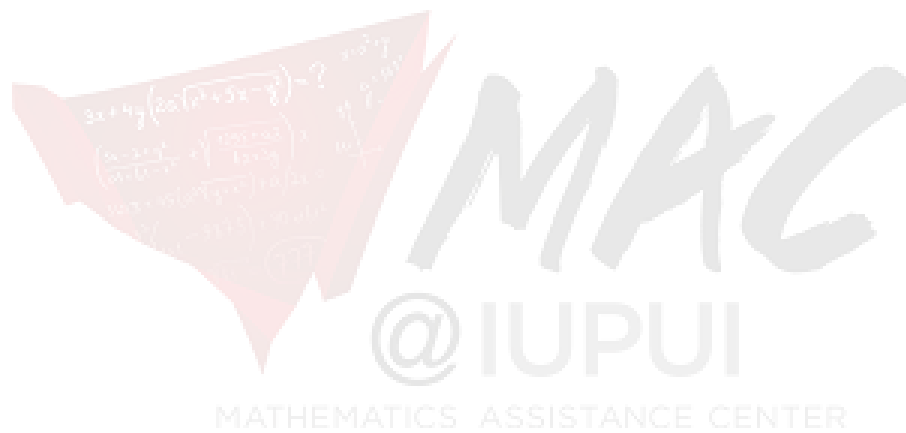
#### 3.1 Inverse Laplace Transformations

Compute the following inverse Laplace transformations.

a.  $\mathcal{L}^{-1}\left(\frac{3}{2s-1}\right)$

b.  $\mathcal{L}^{-1}\left(\frac{s+4}{s^2+4}\right)$

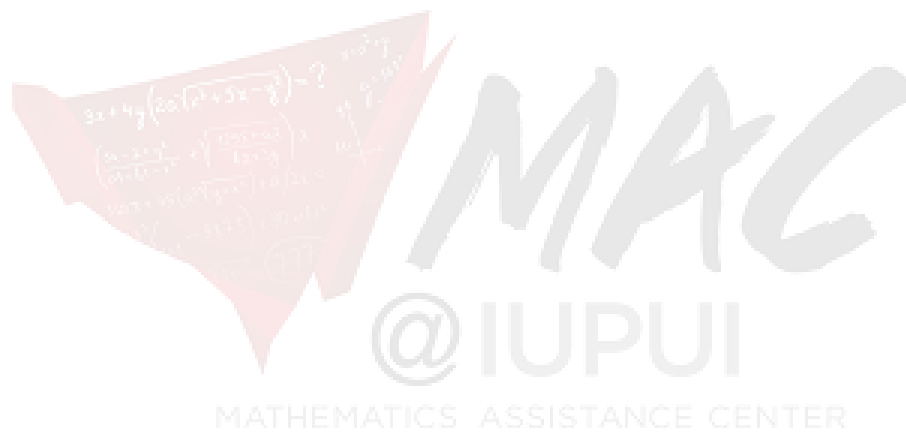
c.  $\mathcal{L}^{-1}\left(\frac{s}{(s+1)(s^2+1)}\right)$



### 3.2 Solutions to Linear Equations

Use the method of Laplace transformations to compute the solution to the given initial-value problem.

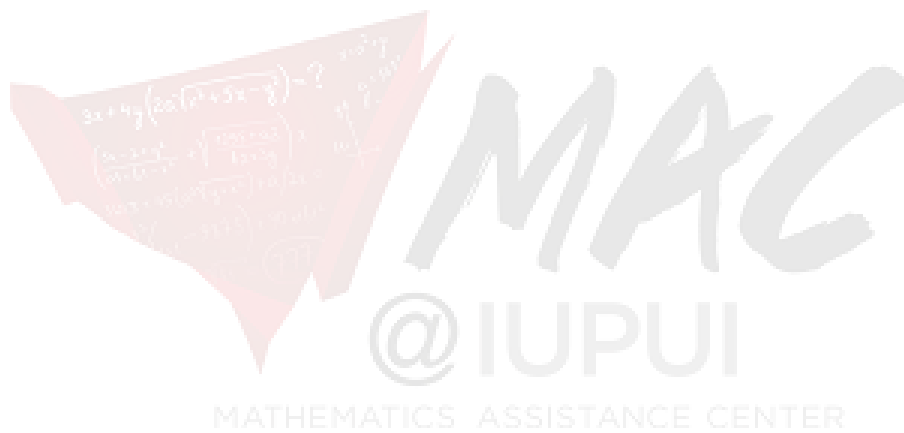
$$\frac{dy}{dt} + 4y = 2 + 4t, \quad y(0) = 1$$





Use the method of Laplace transformations to compute the solution to the given initial-value problem.

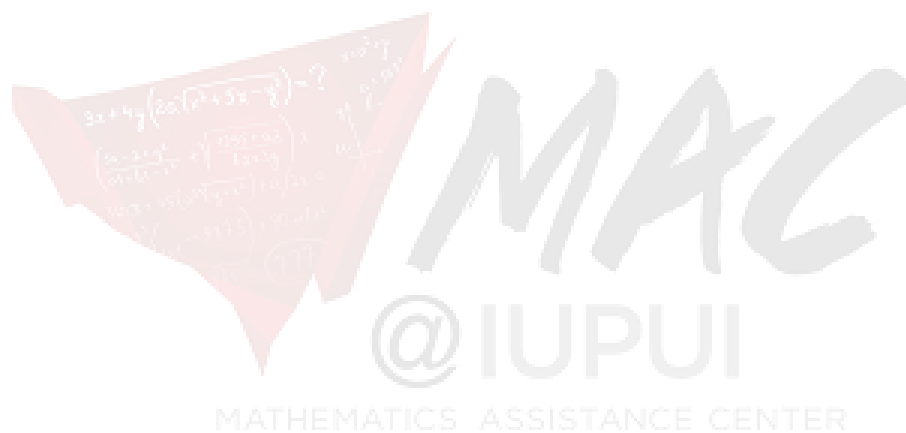
$$\frac{d^2y}{dt^2} + 4y = \cos 5t, \quad y(0) = 0, \quad y'(0) = -2$$



### 3.3 Discontinuous Functions

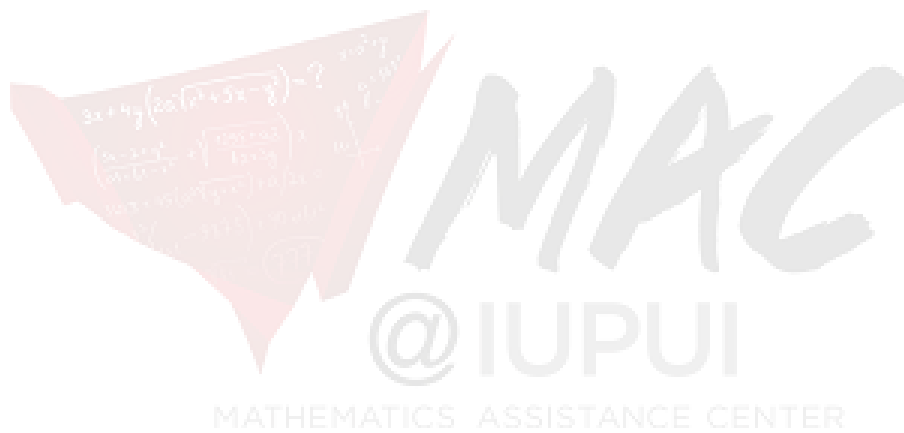
Solve the initial-value problem.

$$\frac{dy}{dt} + y = u_2(t)e^{-2(t-2)}, \quad y(0) = 0$$



Solve the initial-value problem.

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = \delta_3(t), \quad y(0) = 1, \quad y'(0) = 1$$



### 3.4 Convolutions

Compute the convolution  $f * g$  and show that  $\mathcal{L}(f * g) = \mathcal{L}(f) \cdot \mathcal{L}(g)$ .

$$f(t) = \cos t, \quad g(t) = u_2(t)$$

