Contents

1 First-Order Differential Equations

1.1 Separation of Variables

Find the general solution to the given differential equations

a. $\dfrac{dy}{dt} = \dfrac{1}{2y+1}$

b. $\dfrac{dy}{dt} = (y^2+1)t$

c. $\dfrac{dy}{dt} = \dfrac{1}{ty+t+y+1}$

1.2 Slope Fields

Choose points (t, y) with $-2 \le t \le 2$ and $-2 \le y \le 2$ and plot part of the slope field determined by the differential equation without the use of technology.

$$\frac{dy}{dt} = 4y^2$$

1.3 Equilibrium Solutions and Phase Lines

Given the differential equation

$$\frac{dy}{dt} = y^2 - 4y - 12,$$

sketch the graphs of the solutions satisfying the following initial conditions.

a. $y(0) = 1$

b. $y(0) = 6$

c. $y(2) = 5$

d. $y(1) = 7$

1.4 Bifurcations

Locate bifurcation values for the one-parameter family of differential equations and draw phase lines to illustrate your results.

$$\frac{dy}{dt} = (y^2-\alpha)(y^2-4)$$

1.5 Linear Differential Equations

Find the general solution to the given differential equations.

a. $\dfrac{dy}{dt} = -4y + 9e^{-t}$

b. $\dfrac{dy}{dt} = \dfrac{1}{2}y + 4e^{t/2}$

1.6 Integrating Factors

Solve the given initial-value problem.

$$t\frac{dy}{dt} - 2y = 2t^3,\ y(-2) = 4$$

2 Systems of Differential Equations

2.1 Damped Harmonic Oscillators

Use the method of undetermined coefficients to find two non-zero solutions that are not multiples of one another.

$$\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 10y = 0$$

2.2 Decoupled Systems

Consider the following system of differential equations.

$$\begin{aligned} \frac{dx}{dt} &= 2x - 8y^2 \\ \frac{dy}{dt} &= -3y \end{aligned}$$

Derive the general solution and find the solution that corresponds to the initial condition $(x_0, y_0) = (0, 1)$.

2.3 Systems of Linear Equations

For the given system of linear equations

$$\begin{aligned} \frac{dx}{dt} &= -2x - y \\ \frac{dy}{dt} &= 2x - 5y, \end{aligned}$$

form the matrix equation that represents it and check that the functions $\vec{Y}_1(t)$ and $\vec{Y}_2(t)$ are solutions. If the two solutions are linearly independent, form the general solution.

$$\begin{aligned} \vec{Y}_1(t) &= \langle e^{-3t} - 2e^{-4t}, e^{-3t} - 4e^{-4t} \rangle \\ \vec{Y}_2(t) &= \langle 2e^{-3t} + e^{-4t}, 2e^{-3t} + 2e^{-4t} \rangle \end{aligned}$$

2.4 Straight-Line Solutions

Solve the initial-value problem.

$$\frac{d\vec{Y}}{dt} = \begin{pmatrix} -4 & 1 \\ 2 & -3 \end{pmatrix} \vec{Y}, \ \vec{Y}(0) = \langle 2, 1 \rangle$$

2.5 Phase Portraits

Sketch the solution curves in the phase plane and the $x(t)$ and $y(t)$ graphs corresponding to the initial-value problem.

$$\frac{d\vec{Y}}{dt} = \begin{pmatrix} -2 & -2 \\ -2 & 1 \end{pmatrix} \vec{Y}, \ \vec{Y}(0) = \langle 3, -1 \rangle$$

2.6 Complex Eigenvalues

Consider the system below.

$$\frac{d\vec{Y}}{dt} = \begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix} \vec{Y}, \ \vec{Y}(0) = \langle 2, 1 \rangle$$

a. Compute the eigenvalues.

b. Determine if the origin is a spiral sink, spiral souce, or center.

c. Determine the natural period and the natural frequency.

d. Determine the direction of the oscillations in the phase plane.

e. Write the general solution and a solution to the initial-value problem.

f. Draw the solution in the phase plane.

2.7 Special Cases

Consider the system below.

$$\frac{d\vec{Y}}{dt} = \begin{pmatrix} -2 & -1 \\ 1 & -4 \end{pmatrix} \vec{Y}, \ \vec{Y}(0) = \langle 1, 0 \rangle$$

a. Compute the eigenvalues and associated eigenvectors.

b. Write the general solution and the solution to the initial-value problem.

c. Draw solutions in the phase plane.

Consider the system below.

$$\frac{d\vec{Y}}{dt} = \begin{pmatrix} 2 & 4 \\ 3 & 6 \end{pmatrix} \vec{Y}, \ \vec{Y}(0) = \langle 1, 0 \rangle$$

a. Compute the eigenvalues and associated eigenvectors.

b. Write the general solution and the solution to the initial-value problem.

c. Draw solutions in the phase plane.

2.8 Second-Order Linear Equations

Find the general solution of the second-order equation and then find the solution to the initial-value problem.

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} - 5y = 0, \ y(0) = 11, \ y'(0) = -7$$

2.9 Trace-Determinant Plane

Consider the one-parameter family of systems of linear differential equations.

$$\frac{d\vec{Y}}{dt} = \begin{pmatrix} \alpha & 1 \\ \alpha & \alpha \end{pmatrix} \vec{Y}$$

a. Sketch the corresponding curve in the trace-determinant plane.

b. Identify bifurcation values.

c. Describe the different types of behaviors exhibited by the system as α increases along the real line.

2.10 Forced Harmonic Oscillators

Find the general solution for each of the given equations.

a. $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 8y = 2e^{-3t}$

b. $\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 10y = e^{-2t}$

2.11 Sinusoidal Forcing

Using the method of complexification, find the general solution for each of the given equations.

a. $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \cos t$

b. $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \sin t$

2.12 Undamped Forcing and Resonance

Consider the equation below.

$$\frac{d^2y}{dt^2} + 6y = \cos 2t$$

a. Compute the general solution of the differential equation.

b. Determine if the observed phenomena corresponds to beating or resonance.

c. Find the frequency and period of the rapid oscillations and beating (if applicable).

d. Draw a rough sketch of a general solution curve.

Consider the equation below.

$$\frac{d^2y}{dt^2} + 9y = \cos 3t$$

a. Compute the general solution of the differential equation.

b. Determine if the observed phenomena corresponds to beating or resonance.

c. Find the frequency and period of the rapid oscillations and beating (if applicable).

d. Draw a rough sketch of a general solution curve.

2.13 Amplitude and Phase of Solutions

Compute the general solution to the equation below in amplitude-phase form.

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 2\cos 3t$$

What is the steady-state solution to this problem?

3 Laplace Transformations

3.1 Inverse Laplace Transformations

Compute the following inverse Laplace transformations.

a. $\mathcal{L}^{-1}\left(\frac{3}{2s-1}\right)$

b. $\mathcal{L}^{-1}\left(\frac{s+4}{s^2+4}\right)$

c. $\mathcal{L}^{-1}\left(\frac{s}{(s+1)(s^2+1)}\right)$

3.2 Solutions to Linear Equations

Use the method of Laplace transformations to compute the solution to the given initial-value problem.

$$\frac{dy}{dt} + 4y = 2 + 4t,\ y(0) = 1$$

Use the method of Laplace transformations to compute the solution to the given initial-value problem.

$$\frac{d^2y}{dt^2} + 4y = \cos 5t,\ y(0) = 0,\ y'(0) = -2$$

3.3 Discontinuous Functions

Solve the initial-value problem.

$$\frac{dy}{dt} + y = u_2(t)e^{-2(t-2)},\ y(0) = 0$$

Solve the initial-value problem.

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = \delta_3(t),\ y(0) = 1,\ y'(0) = 1$$

3.4 Convolutions

Compute the convolution $f * g$ and show that $\mathcal{L}(f * g) = \mathcal{L}(f) \cdot \mathcal{L}(g)$.

$$f(t) = \cos t,\ g(t) = u_2(t)$$