# Contents

# 1 Vector Functions and Space Curves

## 1.1 Limits, Derivatives, and Integrals of Vector Functions

Consider the vector function

$$\vec{r}(t) = \left\langle \sqrt{4-t^2}, e^{-3t}, \ln(t+1) \right\rangle.$$

a. Find the domain of $\vec{r}(t)$.

b. Compute $\lim_{t\to -1^+} \vec{r}(t)$. Is $\vec{r}(t)$ continuous at $t=-1$?

c. Find $\vec{r}\,'(t)$.

d. Find parametric equations for the tangent line to $\vec{r}(t)$ at $t=0$.

## Solution

a. The domain of $\vec{r}(t)$ is given by

$$(-1, 2]$$

by looking at the domain for each component.

b. For the limit in question to exist, the limit of each component must exist. Since

$$\lim_{t\to -1^+} \ln(t+1)$$

does not exist, the limit in question does not exist! Since the limit at a point must exist for $\vec{r}(t)$ to be continuous at that point, $\vec{r}(t)$ is not continuous at $t=-1$.

c. Compute the derivative component-wise.

$$\vec{r}\,'(t) = \left\langle \frac{-t}{\sqrt{4-t^2}}, -3e^{-3t}, \frac{1}{t+1} \right\rangle$$

d. To find the parametric equations, we first need the value of $\vec{r}(0)$ and $\vec{r}\,'(0)$. We compute them to be

$$\vec{r}(0) = \langle 2, 1, 0\rangle$$
$$\vec{r}\,'(0) = \langle 0, -3, 1\rangle.$$

We now write the vector equation for the tangent line.

$$\vec{l}(t) = \langle 2,1,0\rangle + t\langle 0,-3,1\rangle = \langle 2, 1-3t, t\rangle$$

The parametric equations for the tangent line are now just the components of this vector equation.

$$x(t) = 2$$
$$y(t) = 1-3t$$
$$z(t) = t$$

### 1.2 Arc Length and Curvature

Consider the vector function

$$\vec{r}(t) = \left\langle \sqrt{2}t, e^t, e^{-t} \right\rangle, \ 0 \leq t \leq 1.$$

a. Compute the length of the curve.

b. Find the unit tangent and unit normal vectors to the curve at $t = 0$.

c. Compute the curvature of the function at $t = 0$.

## Solution

a. First, we will need to compute $\vec{r}'(t)$ so we can find $|\vec{r}'(t)|$.

$$\vec{r}'(t) = \langle \sqrt{2}, e^t, -e^{-t} \rangle$$

$$|\vec{r}'(t)| = \sqrt{(\sqrt{2})^2 + (e^t)^2 + (-e^{-t})^2} = \sqrt{e^{2t} + 2 + e^{-2t}} = \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t}$$

The length of the curve is then computed as follows.

$$L = \int_0^1 |\vec{r}'(t)|\, dt = \int_0^1 e^t + e^{-t}\, dt = (e^t - e^{-t})\Big|_0^1 = e - \frac{1}{e}$$

b. Use the formulas for the unit tangent and unit normal vectors.

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}, \ \vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$$

Use the formulas for $\vec{r}'(t)$ and $|\vec{r}'(t)|$ from above to first compute $\vec{T}(t)$.

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{\left\langle \sqrt{2}, e^t, -e^{-t} \right\rangle}{e^t + e^{-t}} = \left\langle \frac{\sqrt{2}}{e^t + e^{-t}}, \frac{e^t}{e^t + e^{-t}}, \frac{-e^{-t}}{e^t + e^{-t}} \right\rangle$$

Now we compute $\vec{T}'(t)$ and $|\vec{T}'(t)|$.

$$\vec{T}'(t) = \left\langle \frac{-\sqrt{2}(e^t - e^{-t})}{(e^t + e^{-t})^2}, \frac{2}{(e^t + e^{-t})^2}, \frac{2}{(e^t + e^{-t})^2} \right\rangle$$

$$|\vec{T}'(t)| = \frac{\sqrt{2}}{e^t + e^{-t}}$$

Now use these formulas to compute $\vec{N}(t)$.

$$\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|} = \left\langle \frac{e^{-t} - e^t}{e^t + e^{-t}}, \frac{\sqrt{2}}{e^t + e^{-t}}, \frac{\sqrt{2}}{e^t + e^{-t}} \right\rangle$$

c. Using $|\vec{T}'(0)| = 1/\sqrt{2}$ and $|\vec{r}'(0)| = 2$, we can compute the curvature at $t = 0$ by

$$\kappa(0) = \frac{|\vec{T}'(0)|}{|\vec{r}'(0)|} = \frac{1/\sqrt{2}}{2} = \frac{1}{2\sqrt{2}}$$

Alternatively, we can use the formula

$$\kappa(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$$

to obtain the same result.

### 1.3 Motion in Space: Velocity and Acceleration

Consider the vector function

$$\vec{r}(t) = \left\langle -\frac{1}{2}t^2, t \right\rangle.$$

a. Find the velocity, acceleration, and speed of the particle with the position function $\vec{r}(t)$.

b. Find the normal and tangent components of the acceleration at $t = 1$.

## Solution

a. Compute the velocity, acceleration, and speed as follows.

$$\begin{aligned}\vec{v}(t) &= \vec{r}'(t) = \langle -t, 1\rangle \\ \vec{a}(t) &= \vec{r}''(t) = \langle -1, 0\rangle \\ v &= |\vec{r}'(t)| = \sqrt{t^2+1}\end{aligned}$$

b. To use the formula given in the book, we first need to compute the curvature.

$$\kappa(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3} = \frac{1}{(t^2+1)^{3/2}}$$

We then have

$$\begin{aligned}\vec{a}_T(1) &= v'(1) = \left.\frac{t}{\sqrt{t^2+1}}\right|_{t=1} = \frac{1}{\sqrt{2}} \\ \vec{a}_N(1) &= \kappa(1)v^2(1) = \frac{1}{2\sqrt{2}}(2) = \frac{1}{\sqrt{2}}.\end{aligned}$$

Alternatively, we may compute the unit tangent and unit normal vectors

$$\vec{T}(1) = \left\langle -1/\sqrt{2}, 1/\sqrt{2} \right\rangle,\ \vec{N}(1) = \left\langle -1/\sqrt{2}, -1/\sqrt{2} \right\rangle$$

and then compute the dot products

$$\begin{aligned}\vec{a}_T(1) &= \vec{a}(1) \cdot \vec{T}(1) = \frac{1}{\sqrt{2}} \\ \vec{a}_N(1) &= \vec{a}(1) \cdot \vec{N}(1) = \frac{1}{\sqrt{2}}\end{aligned}$$

# 2 Functions of Several Variables

## 2.1 Domain and Range

Find and sketch the domain of the function $f(x, y, z)$.

$$f(x, y, z) = \sqrt{1 - x^2 - y^2 - z^2}$$

## Solution

The domain is given by

$$\{(x, y, z) |\, x^2 + y^2 + z^2 \leq 1\}$$

which is the entire unit ball.

### 2.2 Limits and Continuity

Is the function $g(x, y)$ defined by

$$g(x,y) = \begin{cases} \dfrac{x^2 - y^2}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

continuous at $(x, y) = (0, 0)$?

## Solution

For the function to be continuous at $(x, y) = (0, 0)$, we must have $\lim_{(x,y)\to(0,0)} g(x, y) = g(0, 0) = 0$. In particular, the limit must first exist. Notice that if we follow along the curve $y = x$, we have

$$\lim_{x\to 0} \frac{x^2 - x^2}{x^2 + x^2} = 0.$$

This is reassuring, however if we follow along the line $y = 2x$, we have

$$\lim_{x\to 0} \frac{x^2 - (2x)^2}{x^2 + (2x)^2} = \lim_{x\to 0} \frac{-3x^2}{5x^2} = \lim_{x\to 0} \frac{-3}{5} = \frac{-3}{5}.$$

Since we have two different values for the limit, the limit does not exist. This then implies that $g(x, y)$ is not continuous at $(x, y) = (0, 0)$.

Note: We only needed to compute the second limit. With the second limit not equal to 0, we can say that even if the limit did exist, it wouldn't be equal to 0 and the function wouldn't be continuous anyway!

## 2.3 Partial Derivatives

Find all first order partial derivatives of the following functions.

a. $z = (2x + 3y)^{10}$

b. $w = ze^{xyz}$

c. $t = \dfrac{e^v}{u + v^2}$

d. $u = x^{y/z}$

# Solution

a. $\dfrac{\partial z}{\partial x} = 10(2x + 3y)^9(2),\ \dfrac{\partial z}{\partial y} = 10(2x + 3y)^9(3)$

b. $\dfrac{\partial w}{\partial x} = yz^2e^{xyz},\ \dfrac{\partial w}{\partial y}xz^2e^{xyz},\ \dfrac{\partial w}{\partial z} = e^{xyz} + xyze^{xyz}$

c. $\dfrac{\partial t}{\partial u} = \dfrac{-e^v}{(u + v^2)^2},\ \dfrac{\partial t}{\partial v} = \dfrac{(u + v^2)e^v - e^v(2v)}{(u + v^2)^2}$

d. $\dfrac{\partial u}{\partial x} = \dfrac{y}{z}x^{y/z-1},\ \dfrac{\partial u}{\partial y} = x^{y/z}\ln x\left(\dfrac{1}{z}\right),\ \dfrac{\partial u}{\partial z} = x^{y/z}\ln x\left(\dfrac{-y}{z^2}\right)$

## 2.4 Tangent Planes and Linear Approximation

Consider the function

$$f(x, y) = 1 + x \ln(xy - 5).$$

a. Explain why $f(x, y)$ is differentiable at $(x, y) = (2, 3)$.

b. Find the linearization $L(x, y)$ of $f(x, y)$ at the point $(2, 3)$ and use it to approximate $(2.01, 2.99)$.

# Solution

a. $f(x, y)$ is differentiable at $(x, y) = (2, 3)$ because the partial derivatives

$$\frac{\partial f}{\partial x} = \ln(xy - 5) + \frac{xy}{xy - 5}, \; \frac{\partial f}{\partial y} = \frac{x^2}{xy - 5}$$

exist and are continuous at $(x, y) = (2, 3)$.

b. The linearization $L(x, y)$ at $(x, y) = (2, 3)$ is defined by

$$L(x, y) = f(2, 3) + \left( \left. \frac{\partial f}{\partial x} \right|_{(2,3)} \right) (x - 2) + \left( \left. \frac{\partial f}{\partial y} \right|_{(2,3)} \right) (y - 3) = 1 + 6(x - 2) + 4(y - 3)$$

### 2.5 The Chain Rule

Use the chain rule to find the partial derivatives $\frac{\partial z}{\partial s}$, $\frac{\partial z}{\partial t}$, and $\frac{\partial z}{\partial u}$ with $z(x,y)$, $x(s,t,u)$ and $y(s,t,u)$ defined below.

$$z = x^4 + x^2y,\ x = s + 2t - u,\ y = s + u^2$$

## Solution

Use the chain rule.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (4x^3 + 2xy)(1) + (x^2)(1)$$
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = (4x^3 + 2xy)(2) + (x^2)(0)$$
$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} = (4x^3 + 2xy)(-1) + (x^2)(2u)$$

From here, we could substitute back our values of $x$, $y$, and $z$ as given above.

### 2.6 Directional Derivatives and the Gradient

Consider the function

$$f(x, y) = \sin(2x + 3y).$$

a. Find the gradient of $f(x, y)$.

b. Find the maximal rate of change of $f(x, y)$ at the point $(-6, 4)$ and the direction in which it occurs.

c. Find the rate of change of $f(x, y)$ at the point $(-6, 4)$ in the direction of $\langle\sqrt{3}, -1\rangle$.

## Solution

a. The gradient of $f$ is given by

$$\vec{\nabla} f = \langle\cos(2x + 3y)(2), \cos(2x + 3y)(3)\rangle$$

b. The maximal rate of change of $f(x, y)$ at $(-6, 4)$ is the magnitude of the gradient at that point.

$$|\vec{\nabla} f(-6, 4)| = |\langle 2, 3\rangle| = \sqrt{13}$$

The direction of this maximal rate of change is the gradient itself at the point, $\vec{\nabla} f(-6, 4) = \langle 2, 3\rangle$.

c. The question is to find the directional derivative of $f$ at the point $(-6, 4)$ in the direction of $\langle\sqrt{3}, -1\rangle$. First, we need the unit vector in the direction of $\langle\sqrt{3}, -1\rangle$. Divide by the norm to find this unit vector is $\vec{u} = \langle\frac{\sqrt{3}}{2}, \frac{-1}{2}\rangle$. Now use the formula for the directional derivative.

$$D_{\vec{u}} f(-6, 4) = \vec{\nabla} f(-6, 4) \cdot \vec{u} = \langle 2, 3\rangle \cdot \left\langle \frac{\sqrt{3}}{2}, \frac{-1}{2} \right\rangle = \sqrt{3} - \frac{3}{2}$$

## 2.7 Minimum and Maximum Values

Find the local maxima, minima, and saddle points of the following function.

$$f(x, y) = e^y(y^2 - x^2)$$

## Solution

We first need to find all critical points. To do this, we look for where both partial derivatives vanish.

$$\frac{\partial f}{\partial x} = -2xe^y = 0$$
$$\frac{\partial f}{\partial y} = e^y(y^2 + 2y - x^2) = 0$$

Since $e^y > 0$, the first equation tells us we must have that $x = 0$. Plugging this into the second equation, we get that $y^2 + 2y = 0$ and hence, $y = 0$ or $y = -2$. Therefore, the only two critical points are $(0, 0)$ and $(0, -2)$.
To determine whether these points are local maxima, minima, or saddle points, we use the $D$-test. First, we need the second partial derivatives at our critical points.

$$f_{xx}(0,0) = -2 \qquad f_{xy}(0,0) = 0 \qquad f_{yy}(0,0) = 2$$
$$f_{xx}(0,-2) = -2e^{-2} \qquad f_{xy}(0,-2) = 0 \qquad f_{yy}(0,-2) = -2e^{-2}$$

Then we have

$$D(0,0) = f_{xx}(0,0)f_{yy}(0,0) - (f_{xy}(0,0))^2 = -4$$
$$D(0,-2) = f_{xx}(0,-2)f_{yy}(0,-2) - (f_{xy}(0,-2))^2 = 4e^{-4}$$

Since $D(0, 0) < 0$, $(0, 0)$ is a saddle point and since $D(0, -2) > 0$, $(0, -2)$ is a relative extremum. Since $f_{xx}(0, -2) < 0$, it must be that $(0, -2)$ is a local maximum.

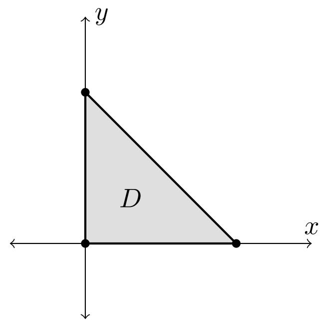
Find the absolute maximum and minimum values of the function

$$f(x, y) = x^2 + y^2 - 2x$$

on the closed, triangular region $D$ with vertices $(2, 0)$, $(0, 2)$, and $(0, -2)$.

## Solution

In the case of a function of one variable, we find critical points on the inside of the interval and then check the endpoints as well. For a function of two variables, we need to find critical points on the inside of the region, critical points along the edges, and then include the corner points. The picture is as follows.




We first compute the critical points inside the triangle by setting the partial derivatives $f_x$ and $f_y$ equal to zero.

$$f_x(x, y) = 2x - 2 = 0,\ f_y(x, y) = 2y = 0 \implies x = 1,\ y = 0$$

Our only critical point is at $(1, 0)$, but this is not on the inside of the triangle so we can ignore it (it will come back when we look for critical points on the bottom edge). We now need to parameterize the sides to find critical points along the edges. For the bottom edge, our parameterization is $x(t) = t$, $y(t) = 0$ for $0 < t < 2$. The function along the bottom curve is then

$$f(x(t), y(t)) = t^2 - 2t.$$

Taking the derivative and setting it equal to zero, we see $t = 1$, or the point $(1, 0)$ is a critical point along the bottom edge. For the right edge, our parameterization is $x(t) = 0$, $y(t) = t$ for $0 < t < 2$. The function along the bottom curve is then

$$f(x(t), y(t)) = t^2$$

which easily has a critical value at $t = 0$, which is the corner $(0, 0)$, so we won't worry about it yet. For the diagonal edge, our parameterization is $x(t) = t$, $y(t) = 2 - t$ for $0 < t < 2$. The function along the diagonal curve is then

$$f(x(t), y(t)) = t^2 + (2 - t)^2 - 2t = 2t^2 - 4t + 1$$

which has a critical point at $t = 1$, or the point $(1, 1)$. Therefore, our critical points are $(1, 0)$, $(1, 1)$, and the corner points $(0, 0)$, $(2, 0)$, and $(0, 2)$. Now we test each of them.

$$f(1, 0) = -1$$
$$f(1, 1) = 0$$
$$f(0, 0) = 0$$
$$f(2, 0) = 0$$
$$f(0, 2) = 4$$

Therefore, the function has an absolute maximum or 4 at $(0, 2)$ and an absolute minimum of $-1$ at $(1, 0)$.

## 2.8 Lagrange Multipliers

Find the extreme values of the function $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

## Solution

At the section title suggests, we will use Lagrange multipliers. Our constraint function is $g(x, y) = x^2+y^2 = 1$. Our system of equation ($\nabla f = \lambda \nabla g$) then becomes

$$\begin{aligned} 2x &= \lambda 2x \\ 4y &= \lambda 2y. \end{aligned}$$

A common mistake at this point is to divide the first equation by $2x$ and get $\lambda = 1$. The mistake here is that we cannot divide by $2x$ if $x = 0$! This means we have to split the problem into cases.
In the case that $x \neq 0$, we can divide by $2x$ and get that $\lambda = 1$. Plugging this into the second equation, we see that $y = 0$. Using this in our constraint, we have

$$x^2 + y^2 = x^2 + 0^2 = 1 \implies x = \pm 1.$$

So we have two critical points, $(1, 0)$ and $(-1, 0)$.
In the case that $x = 0$, we can immediately go to our constraint and see

$$x^2 + y^2 = 0^2 + y^2 = 1 \implies y = \pm 1.$$

So we have two more critical points, $(0, 1)$ and $(0, -1)$.
We now test our critical points.

$$\begin{aligned} f(1, 0) &= 1 \\ f(-1, 0) &= 1 \\ f(0, 1) &= 2 \\ f(0, -1) &= 2 \end{aligned}$$

Therefore, $f$ has an absolute maximum of 2 at $(0, 1)$ and $(0, -1)$ and an absolute minimum of 1 at $(1, 0)$ and $(-1, 0)$.

# 3 Multiple Integrals

## 3.1 Iterated Integrals

Calculate the following iterated integrals.

a. $\displaystyle\int_0^5 12x^2y^3\,dx$

b. $\displaystyle\int_0^2\int_0^4 y^2e^{2x}\,dy\,dx$

c. $\displaystyle\int_1^3\int_1^5 \frac{\ln y}{xy}\,dy\,dx$

d. $\displaystyle\int_{-3}^3\int_0^1 \frac{xy^2}{x^2+1}\,dy\,dx$

## Solution

a. $\displaystyle\int_0^5 12x^2y^3\,dx = 4x^3y^3\Big|_0^5 = 500y^3$

b. $\displaystyle\int_0^2\int_0^4 y^2e^{2x}\,dy\,dx = \int_0^2\left(\frac{1}{3}y^3e^{2x}\Big|_0^4\right)dx = \int_0^2 \frac{64}{3}e^{2x}\,dx = \frac{32}{3}e^{2x}\Big|_0^2 = \frac{32}{3}(e^4-1)$

c. $\displaystyle\int_1^3\int_1^5 \frac{\ln y}{xy}\,dy\,dx = \int_1^3\left(\frac{(\ln y)^2}{2x}\Big|_1^5\right)dx = \int_1^3 \frac{(\ln 5)^2}{2x}\,dx = \frac{1}{2}(\ln 5)^2\ln x\Big|_1^3 = \frac{1}{2}(\ln 5)^2\ln 3$

d. $\displaystyle\int_{-3}^3\int_0^1 \frac{xy^2}{x^2+1}\,dy\,dx = \int_{-3}^3\left(\frac{xy^3}{3(x^2+1)}\Big|_0^1\right)dx = \int_{-3}^3 \frac{x}{3(x^2+1)}\,dx = \frac{3}{2}\ln(x^2+1)\Big|_{-3}^3 = 0$

### 3.2 Double Integrals over General Regions

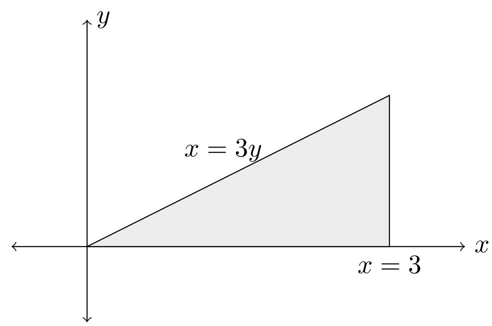
Evaluate the integrals by reversing the oder of integration.

a. $\int_0^1 \int_{3y}^3 e^{x^2}\,dx\,dy$

b. $\int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4}\,dx\,dy$

## Solution

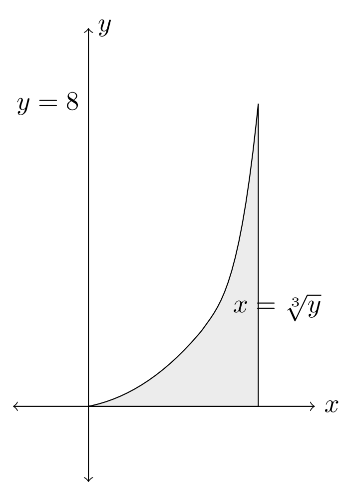
a. First, draw the area of integration.




To switch the order of integration, notice that our $y$-values vary from $y = 0$ up to $3y = x$, or $y = \frac{1}{3}x$. Then we see our $x$-values vary from $x = 0$ to $x = 3$. Our double integral becomes

$$\int_0^1 \int_{3y}^3 e^{x^2}\,dx\,dy = \int_0^3 \int_0^{\frac{1}{3}x} e^{x^2}\,dy\,dx = \int_0^3 \left( ye^{x^2}\Big|_0^{\frac{1}{3}x} \right) dx = \int_0^3 \frac{1}{3}xe^{x^2}\,dx = \frac{1}{6}e^{x^2}\Big|_0^3 = \frac{1}{6}(e^9 - 1).$$

b. Once again, draw the area of integration.




To switch the order of integration, notice that our $y$-values vary from $y = 0$ up to $x = \sqrt[3]{y}$, or $y = x^3$. Our $x$-values then vary from $x = 0$ to $x = 2$. Our double integral becomes

$$\int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4}\,dx\,dy = \int_0^2 \int_0^{x^3} e^{x^4}\,dy\,dx = \int_0^2 x^3 e^{x^4}\,dx = \frac{1}{4}e^{x^4}\Big|_0^2 = \frac{1}{4}(e^{16} - 1)$$

### 3.3 Double Integrals in Polar Coordinates

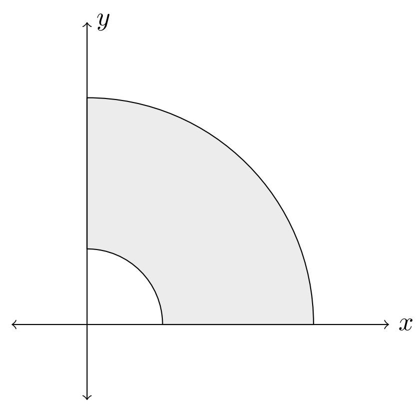
Evaluate the given integral by changing to polar coordinates.

$$\iint_R \sin(x^2 + y^2)\, dA$$

Where $R$ is the region in the first quadrant between the circles of radii 1 and 3 centered at the origin.

## Solution

First, always draw a picture.




From this, we see that our bounds are easily given by $1 \le r \le 3$ and $0 \le \theta \le \pi/2$. Our double integral becomes the following (don't forget the extra $r$!).

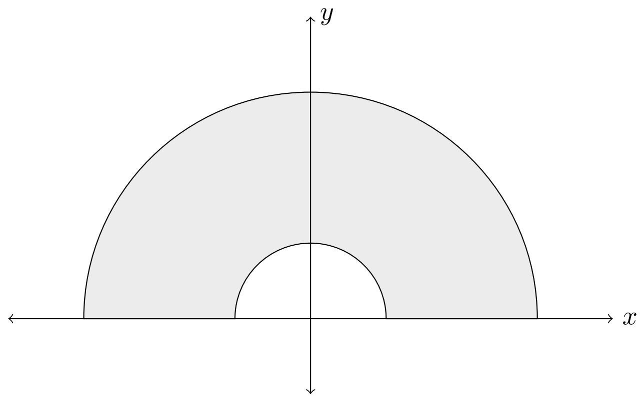
$$\iint_R \sin(x^2 + y^2)\, dA = \int_0^{\pi/2} \int_1^3 r \sin r^2 \, dr\, d\theta = \int_0^{\pi/2} \left( -\frac{1}{2} \cos r^2 \Big|_1^3 \right) d\theta = \frac{\pi}{4}(\cos 1 - \cos 9)$$

### 3.4 Applications of Iterated Integrals

Find the center of mass of the lamina whose boundary consists of the semi-circle $y = \sqrt{1-x^2}$ and $y = \sqrt{4-x^2}$ together with the portions of the $x$-axis that join them and whose density at any point is inversely proportional to its distance from the origin.

## Solution

First, we draw a picture of the area in question.




We first need to calculate the area of this region. This can be done easily in polar coordinates (or using the fact that they are half-circles).

$$A = \int_0^{\pi} \int_1^3 r\, dr\, d\theta = 4\pi$$

Next, use the formulas for center of mass.

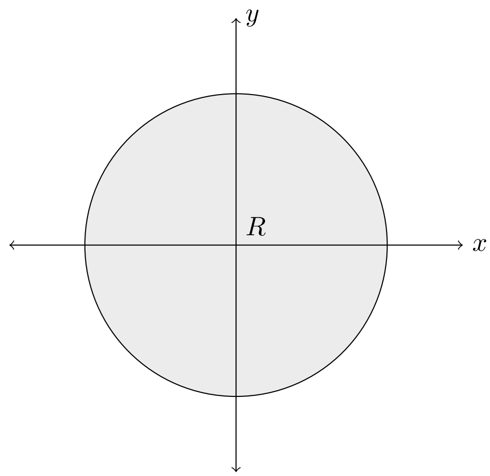
$$\overline{x} = \frac{1}{A} \iint x\, dA = \frac{1}{8\pi} \int_0^{\pi} \int_1^3 (r\cos\theta) r\, dr\, d\theta = \frac{1}{8\pi} \int_0^{\pi} \cos\theta\, d\theta \int_1^3 r^2\, dr = 0$$
$$\overline{y} = \frac{1}{A} \iint y\, dA = \frac{1}{8\pi} \int_0^{\pi} \int_1^3 (r\sin\theta) r\, dr\, d\theta = \frac{1}{8\pi} \int_0^{\pi} \sin\theta\, d\theta \int_1^3 r\, dr = \frac{1}{\pi}$$

Therefore, the center of mass is given by $(\overline{x}, \overline{y}) = (0, 1/\pi)$.

Find the area of the part of the surface $z = xy$ that lies within the cylinder $x^2 + y^2 = 1$.

## Solution

A picture here would help, but drawing this would be difficult. Notice that in the $xy$-plane, the region inside the cylinder $x^2 + y^2 = 1$ is just the unit disk.




Using the formula for surface area of a function, we have the following (using polar coordinates).

$$SA = \iint_R \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA = \iint_R \sqrt{1 + x^2 + y^2}\, dA = \int_0^{2\pi} \int_0^1 r\sqrt{1 + r^2}\, dr\, d\theta = \frac{2\pi}{3} 2^{3/2}$$

### 3.5 Triple Integrals

Evaluate the triple integrals.

a. $\iiint_E y\,dV,\ E = \{(x,y,z)|0 \le x \le 3,\ 0 \le y \le x,\ x-y \le z \le x+y\}$

b. $\iiint_E \frac{z}{x^2+z^2}\,dV,\ E = \{(x,y,z)|1 \le y \le 4,\ y \le z \le 4,\ 0 \le x \le z\}$

## Solution

a. $$\iiint_E y\,dV = \int_0^3 \int_0^x \int_{x-y}^{x+y} y\,dz\,dy\,dx = \int_0^3 \int_0^x yz\Big|_{x-y}^{x+y} dy\,dx = \int_0^3 \int_0^x 2y^2\,dy\,dx = \int_0^3 \frac{2}{3}y^3\Big|_0^x dx$$
$$= \int_0^3 \frac{2}{3}x^3\,dx = \frac{1}{6}x^4\Big|_0^3 = \frac{27}{2}$$

b. $$\iiint_E \frac{z}{x^2+z^2}\,dV = \int_1^4 \int_y^4 \int_0^z \frac{z}{x^2+z^2}\,dx\,dz\,dy = \int_1^4 \int_y^4 \tan^{-1}\left(\frac{x}{z}\right)\Big|_0^z dz\,dy = \int_1^4 \int_y^4 \frac{\pi}{4}\,dz\,dy$$
$$= \int_1^4 \frac{\pi}{4}z\Big|_y^4 dy = \int_1^4 \pi - \frac{\pi}{4}y\,dy = \frac{9\pi}{8}$$

### 3.6 Triple Integrals in Cylindrical Coordinates

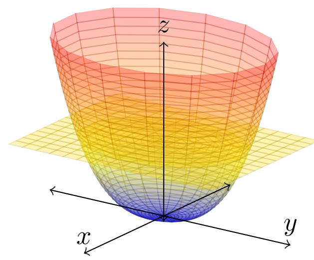
Evaluate the triple integral

$$\iiint_E z\,dV$$

where $E$ is the area enclosed by $z = x^2 + y^2$ and $z = 4$.

## Solution

Always start by drawing a picture.




As the section name suggests, we use cylindrical coordinates to parameterize our region. Our $z$ bound is simple from the picture, $r^2 \leq z \leq 4$. The region in question projected onto the $xy$-plane is just the circle of radius 2, so our $r$ and $\theta$ bounds are $0 \leq \theta \leq 2\pi$ and $0 \leq r \leq 2$. The integral is then given by

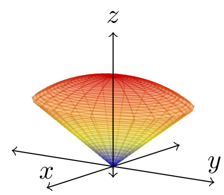
$$\iiint_E z\,dV = \int_0^{2\pi} \int_0^2 \int_{r^2}^4 zr\,dz\,dr\,d\theta = 29\pi$$

### 3.7 Triple Integrals in Spherical Coordinates

Find the volume of the solid that lies below $z = \sqrt{4 - x^2 - y^2}$ and $z = \sqrt{x^2 + y^2}$, and above $z = 0$.

## Solution

We begin by drawing a picture.




From this, we see our bounds to be $0 \leq \rho \leq 2$, $0 \leq \phi \leq \pi/4$, and $0 \leq \theta \leq 2\pi$. The volume is then given by

$$V = \iiint_E dV = \int_0^{2\pi} \int_0^{\pi/4} \int_0^2 \rho^2 \sin\phi \, d\rho \, \phi \, d\theta = \frac{16\pi}{3} \left(1 - \frac{1}{\sqrt{2}}\right)$$

### 3.8 Change of Variables in Iterated Integrals

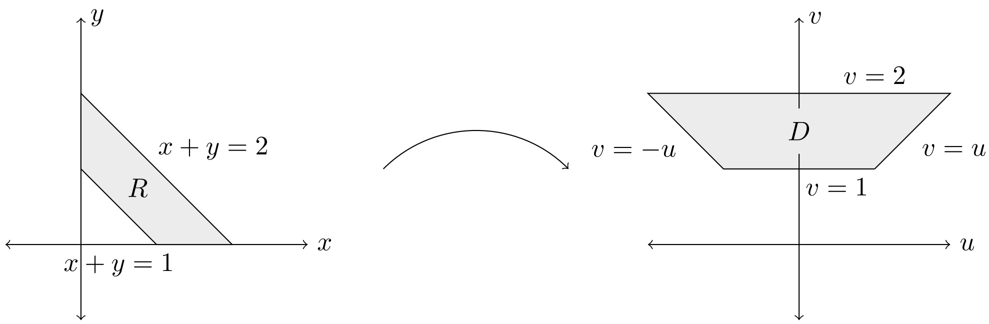
Evaluate the double integral by making an appropriate change of variables.

$$\iint_R \cos\left(\frac{y-x}{y+x}\right)\, dA$$

Where $R$ is the trapezoidal region with vertices $(1,0)$, $(2,0)$, $(0,2)$, and $(0,1)$.

## Solution

The argument of the cosine hints that we should try the change of variables $u = y - x$ and $v = y + x$. The transformation does the following to our bounds (plugging the equations of the lines into our equations for $u$ and $v$).




We now compute our Jacobian (this requires solving for $x$ and $y$ in terms of $u$ and $v$).

$$J = \left|\det\begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}\right| = \left|\det\begin{pmatrix} \frac{-1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}\right| = \left|\frac{-1}{2}\right| = \frac{1}{2}$$

Now we set up our new integral. Note that we integrate with respect to $u$ first, because of the shape of the region $D$.

$$\iint_R \cos\left(\frac{y-x}{y+x}\right)\, dA = \int_1^2 \int_{-v}^{v} \cos\left(\frac{u}{v}\right)\, du\, dv = 3\sin 1$$

# 4 Vector Calculus

## 4.1 Vector Fields

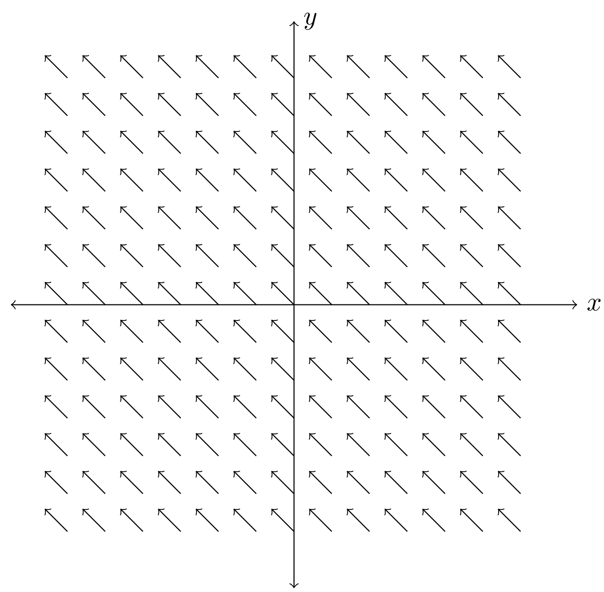
Sketch the vector fields.

a. $\vec{F}(x,y) = -\vec{i} + \vec{j}$

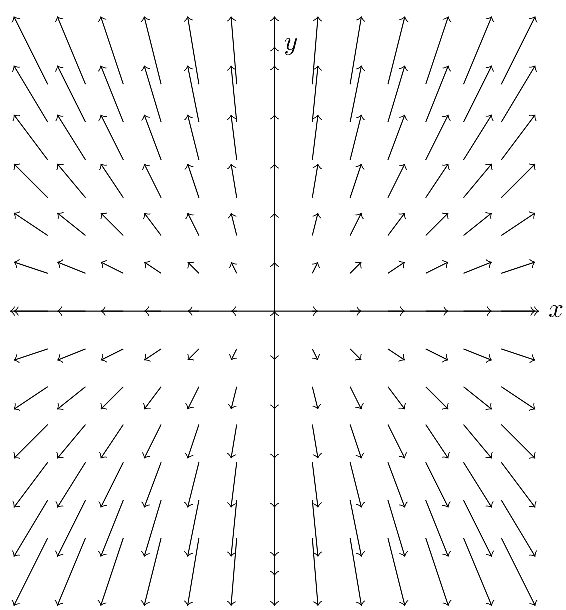
b. $\vec{F}(x,y) = \left\langle \frac{1}{2}x, y \right\rangle$

## Solution

a. The vector fields looks as follows.




b. The vector field looks as follows.



## 4.2 Line Integrals

Calculate the following line integrals.

a. $\int_C xyz^2\, ds$ where $C$ is the line segment from $(-1, 5, 0)$ to $(1, 6, 4)$.

b. $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x, y) = \langle y, x \rangle$ and $C$ is the upper half of the circle of radius 1 centered at the origin oriented counter-clockwise.

## Solution

a. First, we parameterize the line segment as $x(t) = -1 + 2t$, $y(t) = 5 + t$, $z(t) = 4t$ for $0 \leq t \leq 1$ (using the parameterization formula for a line segment). Now we compute the differential $ds$.

$$ds = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt = \sqrt{2^2 + 1^2 + 4^2}\, dt = \sqrt{21}\, dt$$

Now we set up and evaluate our integral as follows.

$$\int_C xyz^2\, ds = \int_0^1 (-1 + 2t)(5 + t)(4t)^2 \sqrt{21}\, dt = 16 \int_0^1 -5t^2 + 9t^3 + 2t^4\, dt = \frac{236}{15}$$

b. We parameterize the upper half of the circle as $\vec{r}(t) = \langle \cos t, \sin t \rangle$ for $0 \leq t \leq \pi$. Now we compute the differential $d\vec{r}$.

$$d\vec{r} = \vec{r}'(t)\, dt = \langle -\sin t, \cos t \rangle\, dt$$

Set up and evaluate the integral as follows.

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^\pi \langle \sin t, \cos t \rangle \cdot \langle -\sin t, \cos t \rangle\, dt = \int_0^\pi \cos^2 t - \sin^2 t\, dt = \int_0^\pi \cos 2t\, dt = 0$$

### 4.3 Fundamental Theorem of Line Integrals

Prove that the vector field $\vec{F}(x,y)$ is conservative and find a function $f$ such that $\vec{\nabla} f = \vec{F}(x,y)$.

$$\vec{F}(x,y) = \langle x^2, y^2 \rangle$$

Use your result to calculate

$$\int_C \vec{F} \cdot d\vec{r}$$

where $C$ is the arc of the parabola $y = 2x^2$ from $(-1,2)$ to $(2,8)$.

## Solution

To prove that the vector field is conservative, we need to find a function $f$ such that $\vec{\nabla} f(x,y) = \vec{F}(x,y)$. This means we need $f(x,y)$ to satisfy

$$\frac{\partial f}{\partial x} = x^2, \;\; \frac{\partial f}{\partial y} = y^2.$$

Integrating the first equation, treating $y$ as a constant, we must have

$$f(x,y) = \frac{1}{3}x^3 + g(y)$$

where $g(y)$ is like our constant of integration. If we differentiate this with respect to $y$, we should get $y^2$, from the second formula. Therefore, we must have

$$\frac{\partial f}{\partial y} = g'(y) = y^2 \implies g(y) = \frac{1}{3}y^3 + c$$

Picking $c = 0$, we can choose

$$f(x,y) = \frac{1}{3}x^3 + \frac{1}{3}y^3.$$

It can easily be checked that $\vec{\nabla} f(x,y) = \langle x^2, y^2 \rangle = \vec{F}(x,y)$. Therefore, $\vec{F}(x,y)$ is conservative. From the fundamental theorem of line integrals, any line integral of $\vec{F}$ is independent of path and the value is the difference of the end points evaluated at $f(x,y)$. Therefore, for the line integral in question, the result is

$$\int_C \vec{F} \cdot d\vec{r} = f(2,8) - f(-1,2) = \left(\frac{1}{3}2^3 + \frac{1}{3}8^3\right) - \left(\frac{1}{3}(-1)^3 + \frac{1}{3}2^3\right) = \frac{513}{3}$$

### 4.4 Green's Theorem

Use Green's Theorem to evaluate the line integral

$$\int_C y^3\,dx - x^3\,dy$$

where $C$ is the circle $x^2 + y^2 = 4$ oriented counterclockwise.

## Solution

Since the functions $y^3$ and $-x^3$ have continuous partial derivatives and the curve $C$ is closed, we can apply Green's theorem to get

$$\int_C y^3\,dx - x^3\,dy = \iint_D -3x^2 - 3y^2\,dA$$

where $D$ is the area inside the circle $x^2 + y^2 = 4$. Switching to polar coordinates, we get

$$\iint_D -3x^2 - 3y^2\,dA = \int_0^{2\pi}\int_0^2 (-3r^2)r\,dr\,d\theta = \int_0^{2\pi}\int_0^2 -3r^3\,dr\,d\theta = -24\pi.$$

Use Green's Theorem to evaluate

$$\int_C \vec{F} \cdot d\vec{r}$$

where

$$\vec{F}(x, y) = \langle y \cos x - xy \sin x, xy + x \cos x \rangle$$

and $C$ is the triangle with vertices $(0, 0)$, $(2, 0)$, and $(0, 4)$.

## Solution

As with the previous problem, since each component of $\vec{F}$ has continuous partial derivatives and $C$ is a closed curve, we can apply Green's theorem.

$$\int_C \vec{F} \cdot d\vec{r} = \iint_D (y + \cos x - x \sin x) - (\cos x - x \sin x) \, dA = \iint_D y \, dA$$

Where $D$ is the triangle determined by the vertices in the problem statement. A picture gives us our bounds.

$$\iint_D y \, dA = \int_0^2 \int_0^{4-2x} y \, dy \, dx = \frac{16}{3}$$

### 4.5 Curl and Divergence

Compute the curl and divergence of the vector field.

$$\vec{F}(x, y, z) = \langle e^x \sin y, e^y \sin z, e^z \sin x \rangle$$

## Solution

The curl and divergence are given by

$$\operatorname{curl} \vec{F} = \langle e^y \cos z, -e^z \cos x, e^x \cos y \rangle$$
$$\operatorname{div} \vec{F} = e^x \sin y + e^y \sin z + e^z \sin x$$

### 4.6 Parametric Surfaces and their Areas

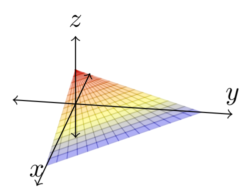
Find the surface area of the part of the plane $3x + 2y + z = 6$ that lies in the first octant.

## Solution

Since we can solve for $z = 6 - 3x - 2y$, we can immediately use the special formula for surface area.

$$SA = \iint_R \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA = \iint_R \sqrt{1 + (-3)^2 + (-2)^2}\, dA = \iint_R \sqrt{14}\, dA$$

Now we need to determine our region of integration, $R$. The picture is as follows.




We see that our $y$-bounds are $y = 0$ to the line where the plane hits $z = 0$, $3x + 2y = 6$, or $y = 3 - \frac{3}{2}x$. Our $x$-bounds are then $x = 0$ to where the line $y = 3 - \frac{3}{2}x$ hits $y = 0$, or $x = 2$. Our integral is then as follows.

$$\int_0^2 \int_0^{3-\frac{3}{2}x} \sqrt{14}\, dA = 3\sqrt{14}$$

Alternatively, notice

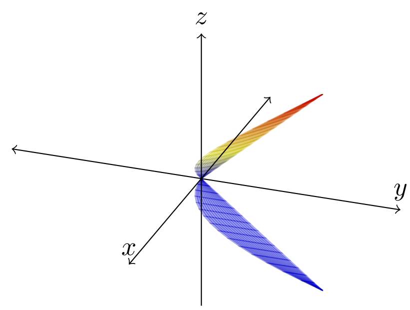
$$SA = \iint_R \sqrt{14}\, dA = \sqrt{14} \iint_R dA = \sqrt{14}\text{Area}(R)$$

Since $R$ is a triangle with base 3 and 2, we have that $\text{Area}(R) = \frac{1}{2}(3)(2) = 3$ and $SA = 3\sqrt{14}$.

Find the surface area of the part of the cone $z = \sqrt{x^2 + y^2}$ that lies between the plane $y = x$ and the cylinder $y = x^2$.

## Solution

As always, draw a picture! The picture is not actually all that difficult or interesting, but it helps us find our region of integration.




From this, we can use the formula from the previous problem with the bounds $0 \leq x \leq 1$ and $x^2 \leq y \leq x$.

$$\int_0^1 \int_{x^2}^x \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dy\, dx = \int_0^1 \int_{x^2}^x \sqrt{1 + \frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2}}\, dy\, dx = \int_0^1 \int_{x^2}^x \sqrt{2}\, dy\, dx = \frac{\sqrt{2}}{6}$$

## Alternative Solution

In general, it can be difficult to solve for $z$ and apply the above formula. In cases like this, the surface should be parameterized. once a parameterization is chosen, $\vec{r}(u, v)$, the differential $dS$ is given by $|\vec{r}_u \times \vec{r}_v|\, dv\, du$ or $|\vec{r}_u \times \vec{r}_v|\, du\, dv$. In this problem, our parameterization could just be $\vec{r}(x, y) = \langle x, y, \sqrt{x^2 + y^2}\rangle$. Our partial derivatives are then

$$\vec{r}_x = \langle 1, 0, \frac{x}{\sqrt{x^2 + y^2}}\rangle$$
$$\vec{r}_y = \langle 0, 1, \frac{y}{\sqrt{x^2 + y^2}}\rangle.$$

Now compute their cross product and its magnitude.

$$\vec{r}_x \times \vec{r}_y = \left\langle \frac{-x}{\sqrt{x^2 + y^2}}, \frac{-y}{\sqrt{x^2 + y^2}}, 1 \right\rangle$$
$$|\vec{r}_x \times \vec{r}_y| = \sqrt{1 + \frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2}} = \sqrt{2}$$

Then out integral is just the same as before!

$$SA = \int_0^1 \int_{x^2}^x \sqrt{2}\, dy\, dx = \frac{\sqrt{2}}{6}$$

Although this way takes a bit more effort, it works in much more generality and is thus preferred!

### 4.7 Surface Integrals

Evaluate the surface integrals.

a. $\iint_S x^2yz\,dS$ where $S$ is the part of the plane $z = 1+2x+3y$ that lies above the rectangle $[0,3]\times[0,2]$.

b. $\iint_S \vec{F}\cdot d\vec{S}$ where $\vec{F}(x,y,z) = \langle x, -z, y\rangle$ and $S$ is the part of the sphere $x^2+y^2+z^2 = 4$ in the first octant with orientation towards the origin.

## Solution

a. As with the previous problem, the trouble here is computing the area of integration and the differential $dS$. Either using the formula from the previous problem or parameterizing the plane by $\vec{r}(x,y) = \langle x, y, 1+2x+3y\rangle$, we find our differential to be

$$dS = |\vec{r}_x \times \vec{r}_y|\,dA = |\langle -2,-3,1\rangle|\,dA = \sqrt{14}\,dA.$$

Our area of integration is given to us in the problem statement, so we set up our integral as follows. (don't forget to substitute in your parameterization if you used one!)

$$\iint_S x^2yz\,dS = \int_0^3\int_0^2 x^2y(1+2x+3y)\sqrt{14}\,dy\,dx = 72\sqrt{14}$$

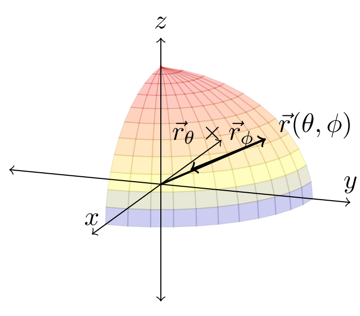
b. This is a case that solving for $z$ explicitly is difficult because we get $z = \pm\sqrt{4-x^2-y^2}$. With two halves of the sphere, we have to do two integrals! Instead of this, we parameterize the sphere as follows (using spherical coordinates).

$$\vec{r}(\theta,\phi) = \langle 2\cos\theta\sin\phi, 2\sin\theta\sin\phi, 2\cos\phi\rangle,\ 0\le\theta\le\frac{pi}{2},\ 0\le\phi\le\frac{\pi}{2}$$

Now we compute our partial derivatives and their cross product.

$$\vec{r}_\theta = \langle -2\sin\theta\sin\phi, 2\cos\theta\sin\phi, 0\rangle$$
$$\vec{r}_\phi = \langle 2\cos\theta\cos\phi, 2\sin\theta\cos\phi, -2\sin\phi\rangle$$
$$\vec{r}_\theta\times\vec{r}_\phi = \langle -4\cos\theta\sin^2\phi, -4\sin\theta\sin^2\phi, -4\sin\phi\cos\phi\rangle = -\sin\phi\langle 4\cos\theta\sin\phi, 4\sin\theta\sin\phi, 4\cos\phi\rangle$$

Note that this is the correct orientation since the normal vector we have is always a negative multiple of the parameterization. The picture is as follows.




Therefore, our differential is given by

$$d\vec{S} = \vec{r}_\theta\times\vec{r}_\phi\,dA = \langle -4\cos\theta\sin^2\phi, -4\sin\theta\sin^2\phi, -4\sin\phi\cos\phi\rangle\,dA.$$

Now, we plug our parameterization into $\vec{F}$.

$$\vec{F}(\vec{r}(\theta,\phi)) = \langle 2\cos\theta\sin\phi, -2\cos\phi, 2\sin\theta\sin\phi\rangle$$

Now we can compute what we're integrating, $\vec{F} \cdot d\vec{S}$.

$$\vec{F} \cdot d\vec{S} = (-8\cos^2\theta\sin^3\phi + 8\sin\theta\sin^2\phi\cos\phi - 8\sin\theta\sin^2\phi\cos\phi)\,dA = -8\cos^2\theta\sin^3\phi\,dA$$

We are integrating over the sphere in the first quadrant, so our bounds are simply $0 \le \theta \le \pi/2$ and $0 \le \phi \le \pi/2$. Our integral is then as follows.

$$\iint_S \vec{F} \cdot d\vec{S} = \int_0^{\pi/2}\int_0^{\pi/2} -8\cos^2\theta\sin^3\phi\,d\phi\,d\theta = -\frac{4\pi}{3}$$

### 4.8 Stoke's Theorem

Use Stoke's Theorem to evaluate

$$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S}$$

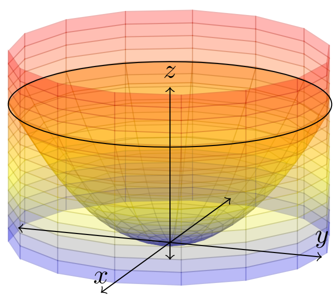
where $\vec{F}(x, y, z) = \langle x^2z^2, y^2z^2, xyz\rangle$ and $S$ is the part of the surface $z = x^2 + y^2$ that lies inside the cylinder $x^2 + y^2 = 4$ oriented upwards.

## Solution

Stokes theorem says that with proper orientation, we have

$$\iint_S \operatorname{curl}\vec{F} \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r}$$

where the curve $C$ is the boundary of the surface $S$ and $C$ is oriented so that the surface $S$ is to the left. In our case, we have the following picture, which tells us that our curve $C$ is oriented counter-clockwise.




From this, we see our boundary is the curve of intersection of the $z = x^2 + y^2$ with $x^2 + y^2 = 4$, which is a circle and can be parameterized by

$$\vec{r}(t) = \langle 2\cos t, 2\sin t, 4\rangle,\ 0 \le t \le 2\pi.$$

Now compute the differential $\vec{F}(\vec{r}(t))$ and $d\vec{r}$.

$$\vec{F}(\vec{r}(t)) = \langle (2\cos t)^2(4)^2, (2\sin t)^2(4)^2, (2\cos t)(2\sin t)(4)\rangle = \langle 64\cos^2 t, 64\sin^2 t, 16\sin t\cos t\rangle$$
$$d\vec{r} = \langle -2\sin t, 2\cos t, 0\rangle\, dt$$

Now we compute what we are integrating, $\vec{F} \cdot d\vec{r}$.

$$\vec{F} \cdot d\vec{r} = (-128\sin t\cos^2 t + 128\sin^2 t\cos t)\, dt$$

Now we set up our integral and evaluate!

$$\int_0^{2\pi} -128\sin t\cos^2 t + 128\sin^2 t\cos t\, dt = 0$$

### 4.9 The Divergence Theorem

Use the Divergence Theorem to calculate the surface integral

$$\iint_S \vec{F} \cdot d\vec{S}$$

where $\vec{F}(x, y, z) = \langle x^3 + y^3, y^3 + z^3, x^3 + z^3 \rangle$ and $S$ is the sphere of radius two centered at the origin.

## Solution

The divergence theorem tells us that with outward orientation on $S$, we have

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \text{div}\vec{F}\, dV.$$

For our purposes, $E$ will be the interior of the sphere of radius two centered at the origin. The divergence of $\vec{F}$ is easily calculated to be

$$\text{div}\vec{F} = 3x^2 + 3y^2 + 3z^2.$$

Using spherical coordinates, we then have that

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \text{div}\vec{F}\, dV = \int_0^{2\pi} \int_0^{\pi} \int_0^{2} (3\rho^2)\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = \frac{382\pi}{5}$$