1 Vector Functions and Space Curves

1.1 Limits, Derivatives, and Integrals of Vector Functions

Consider the vector function
\[ \vec{r}(t) = \left( \sqrt{4-t^2}, e^{-3t}, \ln(t+1) \right). \]

a. Find the domain of \( \vec{r}(t) \).

b. Compute \( \lim_{t \to -1^+} \vec{r}(t) \). Is \( \vec{r}(t) \) continuous at \( t = -1 \)?

c. Find \( \vec{r}'(t) \).

d. Fine parametric equations for the tangent line to \( \vec{r}(t) \) at \( t = 0 \).

Solution

a. The domain of \( \vec{r}(t) \) is given by
\((-1, 2]\)

b. For the limit in question to exist, the limit of each component must exist. Since
\[ \lim_{t \to -1^+} \ln(t+1) \]
does not exist, the limit in question does not exist! Since the limit at a point must exist for \( \vec{r}(t) \) to be continuous at that point, \( \vec{r}(t) \) is not continuous at \( t = -1 \).

c. Compute the derivative component-wise.
\[ \vec{r}'(t) = \left( \frac{-t}{\sqrt{4-t^2}}, -3e^{-3t}, \frac{1}{t+1} \right) \]

d. To find the parametric equations, we first need the value of \( \vec{r}(0) \) and \( \vec{r}'(0) \). We compute them to be
\[ \vec{r}(0) = (2, 1, 0) \]
\[ \vec{r}'(0) = (0, -3, 1). \]

We now write the vector equation for the tangent line.
\[ \vec{l}(t) = (2, 1, 0) + t(0, -3, 1) = (2, 1-3t, t) \]

The parametric equations for the tangent line are now just the components of this vector equation.
\[ x(t) = 2 \]
\[ y(t) = 1 - 3t \]
\[ z(t) = t \]
1.2 Arc Length and Curvature

Consider the vector function

\[ \vec{r}(t) = \left( \sqrt{2}t, e^t, e^{-t} \right), \quad 0 \leq t \leq 1. \]

a. Compute the length of the curve.

b. Find the unit tangent and unit normal vectors to the curve at \( t = 0 \).

c. Compute the curvature of the function at \( t = 0 \).

Solution

a. First, we will need to compute \( \vec{r}'(t) \) so we can find \( |\vec{r}'(t)| \).

\[ \vec{r}'(t) = \left( \sqrt{2}, e^t, -e^{-t} \right) \]

\[ |\vec{r}'(t)| = \sqrt{(\sqrt{2})^2 + (e^t)^2 + (-e^{-t})^2} = \sqrt{2 + e^{2t} + e^{-2t}} = \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t} \]

The length of the curve is then computed as follows.

\[ L = \int_{0}^{1} |\vec{r}'(t)| \, dt = \int_{0}^{1} (e^t + e^{-t}) \, dt = (e^t - e^{-t}) \bigg|_{0}^{1} = e - \frac{1}{e} \]

b. Use the formulas for the unit tangent and unit normal vectors.

\[ \vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}, \quad \vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|} \]

Use the formulas for \( \vec{r}'(t) \) and \( |\vec{r}'(t)| \) from above to first compute \( \vec{T}(t) \).

\[ \vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \left( \frac{\sqrt{2}, e^t, -e^{-t}}{e^t + e^{-t}} \right) = \left( \frac{\sqrt{2}}{e^t + e^{-t}}, \frac{e^t}{e^t + e^{-t}}, \frac{-e^{-t}}{e^t + e^{-t}} \right) \]

Now we compute \( \vec{T}'(t) \) and \( |\vec{T}'(t)| \).

\[ \vec{T}'(t) = \left( \frac{-\sqrt{2}(e^t - e^{-t})}{(e^t + e^{-t})^2}, \frac{2}{(e^t + e^{-t})^2}, \frac{2}{(e^t + e^{-t})^2} \right) \]

\[ |\vec{T}'(t)| = \frac{\sqrt{2}}{e^t + e^{-t}} \]

Now use these formulas to compute \( \vec{N}(t) \).

\[ \vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|} = \left( \frac{e^t - e^{-t}}{e^t + e^{-t}}, \frac{\sqrt{2}}{e^t + e^{-t}}, \frac{-\sqrt{2}}{e^t + e^{-t}} \right) \]

c. Using \( |\vec{T}'(0)| = 1/\sqrt{2} \) and \( |\vec{r}'(0)| = 2 \), we can compute the curvature at \( t = 0 \) by

\[ \kappa(0) = \frac{|\vec{T}'(0)|}{|\vec{r}'(0)|} = \frac{1/\sqrt{2}}{2} = \frac{1}{2\sqrt{2}} \]

Alternatively, we can use the formula

\[ \kappa(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3} \]

to obtain the same result.
1.3 Motion in Space: Velocity and Acceleration

Consider the vector function
\[ \mathbf{r}(t) = \left\langle -\frac{1}{2}t^2, t \right\rangle. \]

a. Find the velocity, acceleration, and speed of the particle with the position function \( \mathbf{r}(t) \).

b. Find the normal and tangent components of the acceleration at \( t = 1 \).

Solution

a. Compute the velocity, acceleration, and speed as follows.

\[
\begin{align*}
\mathbf{v}(t) &= \mathbf{r}'(t) = (-t, 1) \\
\mathbf{a}(t) &= \mathbf{r}''(t) = (-1, 0) \\
v &= |\mathbf{r}'(t)| = \sqrt{t^2 + 1}
\end{align*}
\]

b. To use the formula given in the book, we first need to compute the curvature.

\[
\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{1}{(t^2 + 1)^{3/2}}
\]

We then have

\[
\begin{align*}
\mathbf{a}_T(1) &= v'(1) = \frac{t}{\sqrt{t^2 + 1}} \bigg|_{t=1} = \frac{1}{\sqrt{2}} \\
\mathbf{a}_N(1) &= \kappa(1)v^2(1) = \frac{1}{2\sqrt{2}}(2) = \frac{1}{\sqrt{2}}.
\end{align*}
\]

Alternatively, we may compute the unit tangent and unit normal vectors

\[
\mathbf{T}(1) = \left\langle -1/\sqrt{2}, 1/\sqrt{2} \right\rangle, \quad \mathbf{N}(1) = \left\langle -1/\sqrt{2}, -1/\sqrt{2} \right\rangle
\]

and then compute the dot products

\[
\begin{align*}
\mathbf{a}_T(1) &= \mathbf{a}(1) \cdot \mathbf{T}(1) = \frac{1}{\sqrt{2}} \\
\mathbf{a}_N(1) &= \mathbf{a}(1) \cdot \mathbf{N}(1) = \frac{1}{\sqrt{2}}
\end{align*}
\]
2 Functions of Several Variables

2.1 Domain and Range

Find and sketch the domain of the function \( f(x, y, z) \).

\[
f(x, y) = \sqrt{1 - x^2 - y^2 - z^2}
\]

Solution

The domain is given by

\[
\{(x, y, z)| x^2 + y^2 + z^2 \leq 1 \}
\]

which is entire unit sphere.
2.2 Limits and Continuity

Is the function \( g(x, y) \) defined by

\[
g(x, y) = \begin{cases} 
    x^2 - y^2 & \text{if } (x, y) \neq (0, 0) \\
    x^2 + y^2 & \text{if } (x, y) = (0, 0)
\end{cases}
\]

continuous at \((x, y) = (0, 0)\)?

Solution

For the function to be continuous at \((x, y) = (0, 0)\), we must have \( \lim_{(x,y) \to (0,0)} g(x, y) = g(0, 0) = 0 \). In particular, the limit must first exist. Notice that if we follow along the curve \( y = x \), we have

\[
\lim_{x \to 0} \frac{x^2 - x^2}{x^2 + x^2} = 0.
\]

This is reassuring, however if we follow along the line \( y = 2x \), we have

\[
\lim_{x \to 0} \frac{x^2 - (2x)^2}{x^2 + (2x)^2} = \lim_{x \to 0} \frac{-3x^2}{5x^2} = \lim_{x \to 0} \frac{-3}{5} = \frac{-3}{5}.
\]

Since we have two different values for the limit, the limit does not exist. This then implies that \( g(x, y) \) is not continuous at \((x, y) = (0, 0)\).

Note: We only needed to compute the second limit. With the second limit not equal to 0, we can say that even if the limit did exist, it wouldn’t be equal to 0 and the function wouldn’t be continuous anyway!
2.3 Partial Derivatives

Find all first order partial derivatives of the following functions.

a. \[ z = (2x + 3y)^{10} \]
   \[ \frac{\partial z}{\partial x} = 10(2x + 3y)^9(2), \quad \frac{\partial z}{\partial y} = 10(2x + 3y)^9(3) \]

b. \[ w = ze^{xyz} \]
   \[ \frac{\partial w}{\partial x} = yz^2e^{xyz}, \quad \frac{\partial w}{\partial y} = xz^2e^{xyz}, \quad \frac{\partial w}{\partial z} = e^{xyz} + xye^{xyz} \]

c. \[ t = \frac{e^v}{u + v^2} \]
   \[ \frac{\partial t}{\partial u} = -\frac{e^v}{(u + v^2)^2}, \quad \frac{\partial t}{\partial v} = \frac{(u + v^2)e^v - e^v(2v)}{(u + v^2)^2} \]

d. \[ u = \frac{x^y}{z} \]
   \[ \frac{\partial u}{\partial x} = \frac{y}{z}x^{y/z-1}, \quad \frac{\partial u}{\partial y} = x^{y/z}\ln x\left(\frac{1}{z}\right), \quad \frac{\partial u}{\partial z} = x^{y/z}\ln x\left(-\frac{y}{z^2}\right) \]

Solution
2.4  Tangent Planes and Linear Approximation

Consider the function

\[ f(x, y) = 1 + x \ln(xy - 5). \]

a. Explain why \( f(x, y) \) is differentiable at \( (x, y) = (2, 3) \).

b. Find the linearization \( L(x, y) \) or \( f(x, y) \) at the point \( (2, 3) \) and use it to approximate \( (2.01, 2.99) \).

Solution

a. \( f(x, y) \) is differentiable at \( (x, y) = (2, 3) \) because the partial derivatives

\[
\frac{\partial f}{\partial x} = \ln(xy - 5) + \frac{xy}{xy - 5}, \quad \frac{\partial f}{\partial y} = \frac{x^2}{xy - 5}
\]

exist and are continuous at \( (x, y) = (2, 3) \).

b. The linearization \( L(x, y) \) at \( (x, y) = (2, 3) \) is defined by

\[
L(x, y) = f(2, 3) + \left( \frac{\partial f}{\partial x} \right)_{(2,3)} (x - 2) + \left( \frac{\partial f}{\partial y} \right)_{(2,3)} (y - 3) = 1 + 6(x - 2) + 4(y - 3)
\]
2.5 The Chain Rule

Use the chain rule to find the partial derivatives $\frac{\partial z}{\partial s}$, $\frac{\partial z}{\partial t}$, and $\frac{\partial z}{\partial u}$ with $z(x, y)$, $x(s, t, u)$ and $y(s, t, u)$ defined below.

$$z = x^4 + x^2y, \quad x = s + 2t - u, \quad y = s + u^2$$

Solution

Use the chain rule.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} = (4x^3 + 2xy)(1) + (x^2)(1)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} = (4x^3 + 2xy)(2) + (x^2)(0)$$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u} = (4x^3 + 2xy)(-1) + (x^2)(2u)$$

From here, we could substitute back our values of $x$, $y$, and $z$ as given above.
2.6 Directional Derivatives and the Gradient

Consider the function

\[ f(x, y) = \sin(2x + 3y). \]

a. Find the gradient of \( f(x, y) \).

b. Find the maximal rate of change of \( f(x, y) \) at the point \((-6, 4)\) and the direction in which it occurs.

c. Find the rate of change of \( f(x, y) \) at the point \((-6, 4)\) in the direction of \((\sqrt{3}, -1)\).

Solution

a. The gradient of \( f \) is given by

\[ \nabla f = (\cos(2x + 3y)(2), \cos(2x + 3y)(3)) \]

b. The maximal rate of change of \( f(x, y) \) at \((-6, 4)\) is the magnitude of the gradient at that point.

\[ |\nabla f(-6, 4)| = |(2, 3)| = \sqrt{13} \]

The direction of this maximal rate of change is the gradient itself at the point, \( \nabla f(-6, 4) = (2, 3) \).

c. The question is to find the directional derivative of \( f \) at the point \((-6, 4)\) in the direction of \((\sqrt{3}, -1)\). First, we need the unit vector in the direction of \((\sqrt{3}, -1)\). Divide by the norm to find this unit vector is \( \vec{u} = \left( \frac{\sqrt{3}}{2}, \frac{-1}{2} \right) \). Now use the formula for the directional derivative.

\[ D_{\vec{u}} f(-6, 4) = \nabla f(-6, 4) \cdot \vec{u} = (2, 3) \cdot \left( \frac{\sqrt{3}}{2}, \frac{-1}{2} \right) = \sqrt{3} - \frac{3}{2} \]
2.7 Minimum and Maximum Values

Find the local maxima, minima, and saddle points of the following function.

\[ f(x, y) = e^y(y^2 - x^2) \]

**Solution**

We first need to find all critical points. To do this, we look for where both partial derivatives vanish.

\[ \frac{\partial}{\partial x} = \]

\[ \frac{\partial}{\partial y} = \]
Find the absolute maximum and minimum values of the function

\[ f(x, y) = x^2 + y^2 - 2x \]

on the closed, triangular region \( D \) with vertices \((2, 0), (0, 2), \) and \((0, -2)\).

**Solution**
2.8 Lagrange Multipliers

Find the extreme values of the function $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

Solution
3 Multiple Integrals

3.1 Iterated Integrals

Calculate the following iterated integrals.

a. $\int_0^5 x^2 y^3 \, dx$

b. $\int_0^4 y^2 e^{2x} \, dy \, dx$

c. $\int_1^3 \int_1^5 \frac{\ln y}{xy} \, dy \, dx$

d. $\int_{-3}^3 \int_0^1 \frac{xy^2}{x^2 + 1} \, dy \, dx$

Solution
3.2 Double Integrals over General Regions

Evaluate the integrals by reversing the order of integration.

a. \[ \int_0^1 \int_{3y}^3 e^{x^2} \, dx \, dy \]

b. \[ \int_0^8 \int_{\sqrt[4]{y}}^2 e^{x^4} \, dx \, dy \]

Solution
3.3 Double Integrals in Polar Coordinates

Evaluate the given integral by changing to polar coordinates.

\[ \int \int_{R} \sin(x^2 + y^2) \, dA \]

Where \( R \) is the region in the first quadrant between the circles of radii 1 and 3 centered at the origin.

Solution
3.4 Applications of Iterated Integrals

Find the center of mass of the lamina whose boundary consists of the semi-circle \( y = \sqrt{1-x^2} \) and \( y = \sqrt{4-x^2} \) together with the portions of the \( x \)-axis that join them and whose density at any point is inversely proportional to its distance from the origin.

Solution
Find the area of the part of the surface $z = xy$ that lies within the cylinder $x^2 + y^2 = 1$.

**Solution**
3.5 Triple Integrals

Evaluate the triple integrals.

a. $\iiint_E y \, dV$, $E = \{(x, y, z) | 0 \leq x \leq 3, \ 0 \leq y \leq x, \ x - y \leq z \leq x + y\}$

b. $\iiint_E \frac{z}{x^2 + z^2} \, dV$, $E = \{(x, y, z) | 1 \leq y \leq 4, \ y \leq z \leq 4, \ 0 \leq x \leq z\}$

Solution
3.6 Triple Integrals in Cylindrical Coordinates

Evaluate the triple integral

$$\iiint_E z \, dV$$

where $E$ is the area enclosed by $z = x^2 + y^2$ and $z = 4$.

Solution
3.7 Triple Integrals in Spherical Coordinates

Find the volume of the solid that lies below \( z = \sqrt{4 - x^2 - y^2} \) and \( z = \sqrt{x^2 + y^2} \), and above \( z = 0 \).

Solution
3.8  Change of Variables in Iterated Integrals

Evaluate the double integral by making an appropriate change of variables.

\[
\int \int_{R} \cos \left( \frac{y - x}{y + x} \right) \, dA
\]

Where \( R \) is the trapezoidal region with vertices \((1, 0), (2, 0), (0, 2), \) and \((0, 1)\).

Solution
4 Vector Calculus

4.1 Vector Fields

Sketch the vector fields.

a. \( \vec{F}(x, y) = -\vec{i} + \vec{j} \)

b. \( \vec{F}(x, y) = \left\langle \frac{1}{2} x, y \right\rangle \)

Solution
4.2 Line Integrals

Calculate the following line integrals.

a. \( \int_C xyz^2 \, ds \) where \( C \) is the line segment from \((-1, 5, 0)\) to \((1, 6, 4)\).

b. \( \int_C \vec{F} \cdot d\vec{r} \) where \( \vec{F}(x, y, z) = \langle x, y, xy \rangle \) and \( C \) is the upper semi-circle of the circle of radius 1 centered at the origin oriented counter-clockwise.

\textbf{Solution}
4.3 Fundamental Theorem of Line Integrals

Prove that the vector field \( \vec{F}(x, y) \) is conservative and find a function \( f \) such that \( \nabla f = \vec{F}(x, y) \).

\[
\vec{F}(x, y) = \langle xy^2, y^2 \rangle
\]

Use your result to calculate

\[
\int_C \vec{F} \cdot d\vec{r}
\]

where \( C \) is the arc of the parabola \( y = 2x^2 \) from \((-1, 2)\) to \((2, 8)\).

Solution
4.4 Green’s Theorem

Use Green’s Theorem to evaluate the line integral

\[ \int_C y^2 \, dx - x^3 \, dy \]

where \( C \) is the circle \( x^2 + y^2 = 4 \).

Solution
Use Green’s Theorem to evaluate
\[ \int_C \vec{F} \cdot d\vec{r} \]
where
\[ \vec{F}(x, y) = (y \cos x - xy \sin x, xy + x \cos x) \]
and \( C \) is the triangle with vertices \((0, 0)\), \((2, 0)\), and \((0, 4)\).

**Solution**
4.5 Curl and Divergence

Compute the curl and divergence of the vector field.

\[ \vec{F}(x, y, z) = \langle e^x \sin y, e^y \sin z, e^z \sin x \rangle \]

Solution
4.6  Parametric Surfaces and their Areas

Find the surface area of the part of the plane $3x + 2y + z = 6$ that lies in the first octant.

Solution
Find the surface area of the part of the cone \( z = \sqrt{x^2 + y^2} \) that lies between the plane \( y = x \) and the cylinder \( y = x^2 \).

Solution
4.7 Surface Integrals

Evaluate the surface integrals.

a. \[ \iint_S x^2yz \, dS \] where \( S \) is the part of the plane \( z = 1 + 2x + 3y \) that lies above the rectangle \([0, 3] \times [0, 2]\).

b. \[ \iint_S \vec{F} \cdot d\vec{S} \] where \( \vec{F}(x, y, z) = (x, -z, y) \) and \( S \) is the part of the sphere \( x^2 + y^2 + z^2 = 4 \) in the first octant with orientation towards the origin.

Solution
4.8 Stoke’s Theorem

Use Stoke’s Theorem to evaluate

\[ \iint_S \text{curl} \vec{F} \cdot d\vec{S} \]

where \( \vec{F}(x, y, z) = (x^2z^2, y^2z^2, xyz) \) and \( S \) is the part of the surface \( z = x^2 + y^2 \) that lies inside the cylinder \( x^2 + y^2 = 4 \) oriented upwards.

Solution
4.9 The Divergence Theorem

Use the Divergence Theorem to calculate the surface integral

\[ \iint_S \vec{F} \cdot d\vec{S} \]

where \( \vec{F}(x, y, z) = (x^3 + y^3, y^3 + z^3, x^3 + z^3) \) and \( S \) is the sphere of radius two centered at the origin.

Solution