Contents

1 Vector Functions and Space Curves

1.1 Limits, Derivatives, and Integrals of Vector Functions

Consider the vector function

$$\vec{r}(t) = \left\langle \sqrt{4-t^2}, e^{-3t}, \ln(t+1) \right\rangle.$$

a. Find the domain of $\vec{r}(t)$.

b. Compute $\lim_{t\to -1^+} \vec{r}(t)$. Is $\vec{r}(t)$ continuous at $t = -1$?

c. Find $\vec{r}\,'(t)$.

d. Find parametric equations for the tangent line to $\vec{r}(t)$ at $t = 0$.

1.2 Arc Length and Curvature

Consider the vector function

$$\vec{r}(t) = \left\langle \sqrt{2}t, e^t, e^{-t} \right\rangle,\ 0 \leq t \leq 1.$$

a. Compute the length of the curve.

b. Find the unit tangent and unit normal vectors to the curve.

c. Compute the curvature of the function at $t = 0$.

1.3 Motion in Space: Velocity and Acceleration

Consider the vector function

$$\vec{r}(t) = \left\langle -\frac{1}{2}t^2, t \right\rangle.$$

a. Find the velocity, acceleration, and speed of the particle with the position function $\vec{r}(t)$.

b. Find the normal and tangent components of the acceleration at $t = 1$.

2 Functions of Several Variables

2.1 Domain and Range

Find and sketch the domain of the function $f(x, y, z)$.

$$f(x, y, z) = \sqrt{1 - x^2 - y^2 - z^2}$$

2.2 Limits and Continuity

Is the function $g(x, y)$ defined by

$$g(x,y) = \begin{cases} \dfrac{x^2 - y^2}{x^2 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

continuous at $(x, y) = (0, 0)$?

2.3 Partial Derivatives

Find all first order partial derivatives of the following functions.

a. $z = (2x + 3y)^{10}$

b. $w = ze^{xyz}$

c. $t = \dfrac{e^v}{u + v^2}$

d. $u = x^{y/z}$

2.4 Tangent Planes and Linear Approximation

Consider the function

$$f(x, y) = 1 + x \ln(xy - 5).$$

a. Explain why $f(x, y)$ is differentiable at $(x, y) = (2, 3)$.

b. Find the linearization $L(x, y)$ of $f(x, y)$ at the point $(2, 3)$ and use it to approximate $(2.01, 2.99)$.

2.5 The Chain Rule

Use the chain rule to find the partial derivatives $\frac{\partial z}{\partial s}$, $\frac{\partial z}{\partial t}$, and $\frac{\partial z}{\partial u}$ with $z(x, y)$, $x(s, t, u)$ and $y(s, t, u)$ defined below.

$$z = x^4 + x^2 y, \ x = s + 2t - u, \ y = s + u^2$$

2.6 Directional Derivatives and the Gradient

Consider the function

$$f(x, y) = \sin(2x + 3y).$$

a. Find the gradient of $f(x, y)$.

b. Find the maximal rate of change of $f(x, y)$ at the point $(-6, 4)$ and the direction in which it occurs.

c. Find the rate of change of $f(x, y)$ at the point $(-6, 4)$ in the direction of $\langle \sqrt{3}, -1 \rangle$.

2.7 Minimum and Maximum Values

Find the local maxima, minima, and saddle points of the following function.

$$f(x, y) = e^y(y^2 - x^2)$$

Find the absolute maximum and minimum values of the function

$$f(x, y) = x^2 + y^2 - 2x$$

on the closed, triangular region D with vertices $(2, 0)$, $(0, 2)$, and $(0, -2)$.

2.8 Lagrange Multipliers

Find the extreme values of the function $f(x, y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

3 Multiple Integrals

3.1 Iterated Integrals

Calculate the following iterated integrals.

a. $\displaystyle\int_0^5 12x^2y^3\,dx$

b. $\displaystyle\int_0^2\int_0^4 y^2e^{2x}\,dy\,dx$

c. $\displaystyle\int_1^3\int_1^5 \frac{\ln y}{xy}\,dy\,dx$

d. $\displaystyle\int_{-3}^3\int_0^1 \frac{xy^2}{x^2+1}\,dy\,dx$

3.2 Double Integrals over General Regions

Evaluate the integrals by reversing the oder of integration.

a. $\displaystyle\int_0^1 \int_{3y}^3 e^{x^2}\,dx\,dy$

b. $\displaystyle\int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4}\,dx\,dy$

3.3 Double Integrals in Polar Coordinates

Evaluate the given integral by changing to polar coordinates.

$$\iint_R \sin(x^2 + y^2)\, dA$$

Where R is the region in the first quadrant between the circles of radii 1 and 3 centered at the origin.

3.4 Applications of Iterated Integrals

Find the center of mass of the lamina whose boundary consists of the semi-circle $y = \sqrt{1-x^2}$ and $y = \sqrt{4-x^2}$ together with the portions of the x-axis that join them and whose density at any point is inversely proportional to its distance from the origin.

Find the area of the part of the surface $z = xy$ that lies within the cylinder $x^2 + y^2 = 1$.

3.5 Triple Integrals

Evaluate the triple integrals.

a. $\iiint_E y\, dV,\ E = \{(x, y, z)|0 \le x \le 3,\ 0 \le y \le x,\ x - y \le z \le x + y\}$

b. $\iiint_E \dfrac{z}{x^2 + z^2}\, dV,\ E = \{(x, y, z)|1 \le y \le 4,\ y \le z \le 4,\ 0 \le x \le z\}$

3.6 Triple Integrals in Cylindrical Coordinates

Evaluate the triple integral

$$\iiint_E z\, dV$$

where E is the area enclosed by $z = x^2 + y^2$ and $z = 4$.

3.7 Triple Integrals in Spherical Coordinates

Find the volume of the solid that lies below $z = \sqrt{4 - x^2 - y^2}$ and $z = \sqrt{x^2 + y^2}$, and above $z = 0$.

3.8 Change of Variables in Iterated Integrals

Evaluate the double integral by making an appropriate change of variables.

$$\iint_R \cos\left(\frac{y-x}{y+x}\right)\, dA$$

Where R is the trapezoidal region with vertices $(1,0)$, $(2,0)$, $(0,2)$, and $(0,1)$.

4 Vector Calculus

4.1 Vector Fields

Sketch the vector fields.

a. $\vec{F}(x, y) = -\vec{i} + \vec{j}$

b. $\vec{F}(x, y) = \left\langle \frac{1}{2}x, y \right\rangle$

4.2 Line Integrals

Calculate the following line integrals.

a. $\int_C xyz^2 \, ds$ where C is the line segment from $(-1, 5, 0)$ to $(1, 6, 4)$.

b. $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x, y) = \langle y, x \rangle$ and C is the upper half of the circle of radius 1 centered at the origin oriented counter-clockwise.

4.3 Fundamental Theorem of Line Integrals

Prove that the vector field $\vec{F}(x, y)$ is conservative and find a function f such that $\vec{\nabla} f = \vec{F}(x, y)$.

$$\vec{F}(x, y) = \langle x^2, y^2 \rangle$$

Use your result to calculate

$$\int_C \vec{F} \cdot d\vec{r}$$

where C is the arc of the parabola $y = 2x^2$ from $(-1, 2)$ to $(2, 8)$.

4.4 Green's Theorem

Use Green's Theorem to evaluate the line integral

$$\int_C y^3\,dx - x^3\,dy$$

where C is the circle $x^2 + y^2 = 4$ oriented counterclockwise.

Use Green's Theorem to evaluate

$$\int_C \vec{F} \cdot d\vec{r}$$

where

$$\vec{F}(x, y) = \langle y \cos x - xy \sin x, xy + x \cos x \rangle$$

and C is the triangle with vertices $(0, 0)$, $(2, 0)$, and $(0, 4)$.

4.5 Curl and Divergence

Compute the curl and divergence of the vector field.

$$\vec{F}(x, y, z) = \langle e^x \sin y, e^y \sin z, e^z \sin x \rangle$$

4.6 Parametric Surfaces and their Areas

Find the surface area of the part of the plane $3x + 2y + z = 6$ that lies in the first octant.

Find the surface area of the part of the cone $z = \sqrt{x^2 + y^2}$ that lies between the plane $y = x$ and the cylinder $y = x^2$.

4.7 Surface Integrals

Evaluate the surface integrals.

a. $\iint_S x^2yz\,dS$ where S is the part of the plane $z = 1 + 2x + 3y$ that lies above the rectangle $[0, 3] \times [0, 2]$.

b. $\iint_S \vec{F} \cdot d\vec{S}$ where $\vec{F}(x, y, z) = \langle x, -z, y \rangle$ and S is the part of the sphere $x^2 + y^2 + z^2 = 4$ in the first octant with orientation towards the origin.

4.8 Stoke's Theorem

Use Stoke's Theorem to evaluate

$$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S}$$

where $\vec{F}(x, y, z) = \langle x^2 z^2, y^2 z^2, xyz \rangle$ and S is the part of the surface $z = x^2 + y^2$ that lies inside the cylinder $x^2 + y^2 = 4$ oriented upwards.

4.9 The Divergence Theorem

Use the Divergence Theorem to calculate the surface integral

$$\iint_S \vec{F} \cdot d\vec{S}$$

where $\vec{F}(x, y, z) = \langle x^3 + y^3, y^3 + z^3, x^3 + z^3 \rangle$ and S is the sphere of radius two centered at the origin.