Contents

1 Matrices

1.1 Matrix Operations

Given the matrices

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} -1 & 1 & 0 \\ 4 & -2 & 2 \end{pmatrix}$$

Compute the following, if they are defined.

1. $\mathbf{AB}$
2. $\mathbf{BA}$
3. $\mathbf{A} + \mathbf{B}$
4. $3\mathbf{A}$

Solution

1. $\mathbf{AB} = \begin{pmatrix} (2)(-1) + (1)(4) & (2)(1) + (1)(-2) & (2)(0) + (1)(2) \\ (-1)(-1) + (3)(4) & (-1)(1) + (3)(-2) & (-1)(0) + (3)(2) \end{pmatrix} = \begin{pmatrix} 2 & 0 & 2 \\ 13 & -7 & 6 \end{pmatrix}$
2. $\mathbf{BA}$ is not defined since $\mathbf{B}$ is a 2×3 matrix and $\mathbf{A}$ is a 2×2 matrix. Since the number of columns of $\mathbf{B}$ is not the same as the number of rows of $\mathbf{A}$, it is not defined.
3. $\mathbf{A} + \mathbf{B}$ is also not defined since $\mathbf{A}$ and $\mathbf{B}$ do not have the same size.
4. $3\mathbf{A} = \begin{pmatrix} 3(2) & 3(1) \\ 3(-1) & 3(3) \end{pmatrix} = \begin{pmatrix} 6 & 3 \\ -3 & 9 \end{pmatrix}$

1.2 Systems of Equations

Find all solutions, if any, to the given system of equations using either Gaussian elimination or Gauss-Jordan elimination.

$$\begin{aligned} 4x + 3y &= 5 \\ 3x - 2y &= 8 \end{aligned}$$

Solution

First, write the system as a matrix.

$$\left(\begin{array}{cc|c} 4 & 3 & 5 \\ 3 & -2 & 8 \end{array}\right)$$

Now use Gaussian elimination.

$$\left(\begin{array}{cc|c} 4 & 3 & 5 \\ 3 & -2 & 8 \end{array}\right) \xrightarrow{R_1 - R_2} \left(\begin{array}{cc|c} 1 & 5 & -3 \\ 3 & -2 & 8 \end{array}\right) \xrightarrow{R_2 - 3R_1} \left(\begin{array}{cc|c} 1 & 5 & -3 \\ 0 & -17 & 17 \end{array}\right) \xrightarrow{-\frac{1}{17}R_2} \left(\begin{array}{cc|c} 1 & 5 & -3 \\ 0 & 1 & -1 \end{array}\right)$$

Now we pull our equations out to get

$$x + 5y = -3, \;\; y = -1$$

Plugging in $y = -1$ into the first equation, we see that $x = 2$. Therefore, the only solution is $x = 2$, $y = -1$.

(Note: To do Gauss-Jordan elimination, there is only one more step needed in the row reduction.)

$$\left(\begin{array}{cc|c} 1 & 5 & -3 \\ 0 & 1 & -1 \end{array}\right) \xrightarrow{R_1 - 5R_2} \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array}\right)$$

This then gives $x = 2$, $y = -1$.

Find all solutions to the following system of equations.

$$\begin{aligned} x + 2z &= 1 \\ x + y + z &= 3 \\ 2x + 3y + 2z &= 9 \end{aligned}$$

Solution

Put the system of equations into a matrix.

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 1 & 1 & 1 & 3 \\ 2 & 3 & 2 & 9 \end{array}\right)$$

Now row reduce.

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 1 & 1 & 1 & 3 \\ 2 & 3 & 2 & 9 \end{array}\right) \xrightarrow{R_2-R_1} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 2 \\ 2 & 3 & 2 & 9 \end{array}\right) \xrightarrow{R_3-2R_1} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 3 & -2 & 7 \end{array}\right) \xrightarrow{R_3-3R_2} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

We may stop here and plug numbers in, we however choose to do Gauss-Jordan elimination and reduce further.

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right) \xrightarrow{R_1-2R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right) \xrightarrow{R_2+R_3} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

Therefore, we have the only solution is $x = -1$, $y = 3$ and $z = 1$.

1.3 Determinants and Inverses

Find the determinants of the following matrices **A** and **B**

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & 0 \\ 2 & -1 & -2 \\ -3 & 3 & 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 4 & 1 & 0 \\ 3 & 2 & 1 \\ 0 & 4 & 2 \end{pmatrix}$$

Solution

Do the cofactor expansion to find $\det \mathbf{A}$.

$$\det \mathbf{A} = (1) \det \begin{pmatrix} -1 & -2 \\ 3 & 1 \end{pmatrix} - (-1) \det \begin{pmatrix} 2 & -2 \\ -3 & 1 \end{pmatrix} + (0) \det \begin{pmatrix} 2 & -1 \\ -3 & 3 \end{pmatrix} = (1)(5) - (-1)(-4) + (0)(3) = 1$$

Do the same to find $\det \mathbf{B}$.

$$\det \mathbf{B} = (4) \det \begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix} - (1) \det \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix} + (0) \det \begin{pmatrix} 3 & 2 \\ 0 & 4 \end{pmatrix} = (4)(0) - (1)(6) + (0)(12) = -6$$

Find the inverses of the following matrices.

$$\mathbf{A} = \begin{pmatrix} 3 & 4 \\ 5 & 6 \end{pmatrix}, \ \mathbf{B} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}$$

Solution 1

To find the inverse matrix of **A**, we row reduce the following matrix.

$$\left(\begin{array}{cc|cc} 3 & 4 & 1 & 0 \\ 5 & 6 & 0 & 1 \end{array}\right)$$

We row reduce as follows.

$$\left(\begin{array}{cc|cc} 3 & 4 & 1 & 0 \\ 5 & 6 & 0 & 1 \end{array}\right) \xrightarrow{2R_1} \left(\begin{array}{cc|cc} 6 & 8 & 2 & 0 \\ 5 & 6 & 0 & 1 \end{array}\right) \xrightarrow{R_1 - R_2} \left(\begin{array}{cc|cc} 1 & 2 & 2 & -1 \\ 5 & 6 & 0 & 1 \end{array}\right) \xrightarrow{R_2 - 5R_1} \left(\begin{array}{cc|cc} 1 & 2 & 2 & -1 \\ 0 & -4 & -10 & 6 \end{array}\right)$$
$$\xrightarrow{-\frac{1}{4}R_2} \left(\begin{array}{cc|cc} 1 & 2 & 2 & -1 \\ 0 & 1 & \frac{5}{2} & -\frac{3}{2} \end{array}\right) \xrightarrow{R_1 - 2R_2} \left(\begin{array}{cc|cc} 1 & 0 & -3 & 2 \\ 0 & 1 & \frac{5}{2} & -\frac{3}{2} \end{array}\right)$$

Then we have that $\mathbf{A}^{-1}$ is the right side of this matrix, or in other words

$$\mathbf{A}^{-1} = \begin{pmatrix} -3 & 2 \\ \frac{5}{2} & -\frac{3}{2} \end{pmatrix}$$

Do the same for **B** to obtain

$$\mathbf{B}^{-1} = \begin{pmatrix} \frac{3}{2} & -\frac{1}{2} \\ 1 & 0 \end{pmatrix}$$

Solution 2

Use the formula

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

This gives us the same results as above.

1.4 Eigenvalues and Eigenvectors

Compute the eigenvalues and eigenvectors for the following matrix.

$$\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}$$

Solution

To compute the eigenvalues, first compute the characteristic equation and set it equal to zero.

$$\det(A - \lambda I) = \det \begin{pmatrix} 1-\lambda & -1 \\ 2 & 4-\lambda \end{pmatrix} = (1-\lambda)(4-\lambda) + 2 = \lambda^2 - 5\lambda + 6 = 0$$

This can be solved by factoring or quadratic formula to show that $\lambda_1 = 2$ and $\lambda_2 = 3$ (the order doesn't matter).

To compute the eigenvectors for $\lambda_1 = 2$, we look for solutions to the equation

$$\begin{pmatrix} 1-\lambda_1 & -1 \\ 2 & 4-\lambda_1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

This can be solved with row reduction (or just by pulling out the equations) to get that $\alpha = -\beta$. We can choose any α and β that work. We choose $\alpha = 1$ and $\beta = -1$. Then our eigenvector is

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

We do the same with λ_2 to get

$$\vec{v}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

1.5 Difference Equations

Solve the following initial value difference equation. (Note: Not all classes cover this material.)

$$x_{n+1} - 5x_n + 6x_{n-1} = 0, \quad x_0 = 4, \quad x_1 = 9$$

Solution

Begin by setting up the characteristic equation.

$$r^2 - 5r + 6 = 0$$

This has solutions $r_1 = 2$ and $r_2 = 3$. Then our solution has the form

$$x_n = c_1 2^n + c_2 3^n$$

If we let $n = 0$, we have the following from our initial data.

$$x_0 = c_1 + c_2 = 4$$

Similarly, if $n = 1$, we see

$$x_1 = 2c_1 + 3c_2 = 9$$

So we arrive at the system of equations.

$$\begin{aligned} c_1 + c_2 &= 4 \\ 2c_1 + 3c_2 &= 9 \end{aligned}$$

Solving by whatever method you wish yields $c_1 = 3$ and $c_2 = 1$. Therefore, our solution is the following.

$$x_n = 3(2^n) + 3^n$$

2 Functions of Several Variables

2.1 Function Values

The surface area of a person (in m^2) can be approximated using a formula known as the Haycock formula. The formula is a function of two variables, the person's height in centimeters and the person's weight in kilograms. The formula is as follows

$$\text{S}(h, w) = .024265h^{.3964}w^{.5378}$$

Estimate the surface area of a person who is 165cm tall and weighs 80kg.

Solution

Plug in the values into the formula.

$$S(165, 80) = .024265 \times (165)^{.3964} \times (80)^{.5378} = 1.9385\text{m}^2$$

2.2 Partial Derivatives

Find first and second order partial derivatives of the function $f(x, y) = 3xy^2 + 2xy + x^2$.

Solution

Take the derivatives desired while keeping other variables constant.

$$\begin{aligned}
f_x(x, y) &= 3y^2 + 2y + 2x \\
f_y(x, y) &= 6xy + 2x \\
f_{xx}(x, y) &= 2 \\
f_{xy}(x, y) &= 6y + 2 \\
f_{yx}(x, y) &= 6y + 2 \\
f_{yy}(x, y) &= 6x
\end{aligned}$$

2.3 Maximum-Minimum Value Problems

Find all relative maximum and minimum values of the function

$$f(x,y) = 4xy - x^3 - y^2$$

and verify your result using the D-Test.

Solution

First, find points where the first partial derivatives are equal to zero.

$$f_x(x,y) = 4y - 3x^2 = 0$$
$$f_y(x,y) = 4x - 2y = 0$$

The second equation tells us that $y = 2x$. Plugging this into the first equation, we have

$$8x - 3x^2 = 0$$

This has solutions of $x = 0$ and $x = 8/3$. Using the fact that $y = 2x$, we have the points

$$(0,0)\,, \quad \left(\frac{8}{3}, \frac{16}{3}\right)$$

Now use the D-test to determine whether these points are relative maximum, relative minimum, or neither. To do so, first find the second partial derivatives.

$$f_{xx}(x,y) = -6x$$
$$f_{xy}(x,y) = 4$$
$$f_{yy}(x,y) = -2$$

(Note: We didn't calculate f_{yx} since it is the same as f_{xy} by Clairaut's Theorem.) Then we have

$$D(x,y) = (f_{xx}(x,y))(f_{yy}(x,y)) - (f_{xy}(x,y))^2 = 12x - 16$$

Now plug in our points from the first part.

$$D(0,0) = -16$$
$$D(8/3, 16/3) = 16$$

Therefore, $(0,0)$ is a local maximum (Since $D < 0$) and $(8/3, 16/3)$ is a local minimum (Since $D > 0$).

2.4 Separable Differential Equations

Find the general solution to the given separable differential equations

1. $\dfrac{dy}{dx} = y \tan x$
2. $\dfrac{1}{\sin x}\dfrac{dy}{dx} = y \cos x$
3. $\dfrac{1}{(\sin x + \cos x)^2}\dfrac{dy}{dx} = y^2$

Solution

1. As the title suggests, separate the variables.

$$\frac{dy}{dx} = y \tan x$$
$$\frac{dy}{y} = \tan x \, dx$$

Now integrate both sides.

$$\int \frac{dy}{y} = \int \tan x \, dx$$
$$\ln |y| = \ln |\sec x| + c$$
$$y = e^{\ln|\sec x| + c}$$
$$y = C \sec x$$

2. Separate variables.

$$\frac{1}{\sin x}\frac{dy}{dx} = y \cos x$$
$$\frac{dy}{y} = \sin x \cos x \, dx$$

And integrate.

$$\int \frac{dy}{y} = \int \sin x \cos x \, dx$$
$$\ln |y| = \frac{1}{2}\sin^2 x + c$$
$$y = Ce^{\frac{1}{2}\sin^2 x}$$

3. Separate variables again.

$$\frac{1}{(\sin x + \cos x)^2}\frac{dy}{dx} = y^2$$
$$\frac{dy}{y^2} = (\sin x + \cos x)^2 \, dx$$

And integrate.

$$\begin{aligned}
\int \frac{dy}{y^2} &= \int (\sin x + \cos x)^2 \, dx \\
-\frac{1}{y} &= \int (1 + 2 \sin x \cos x) \, dx \\
-\frac{1}{y} &= x + \sin^2 x \\
y &= \frac{-1}{x + \sin^2 x}
\end{aligned}$$

3 First Order Differential Equations

3.1 First Order Linear Differential Equations

Find the general solution to the given differential equations

1. $y' + 2y = e^t$
2. $y' + \frac{2}{x}y = \frac{2e^{x^2}}{x}$
3. $y' + y\tan x = \cos x$

Solution

1. First, find the homogeneous solution by the characteristic equation.

$$r + 2 = 0$$

Which only has the solution $r = -2$. Therefore, the homogeneous solution is

$$y_h(t) = ce^{-2t}$$

Now we worry about our particular solution. To find this solution, we make the guess

$$y_p(t) = Ae^t$$

Then

$$y_p'(t) = Ae^t$$

Plugging these into our equation, we get

$$\begin{aligned} Ae^t + 2Ae^t &= e^t \\ 3Ae^t &= e^t \\ 3A &= 1 \\ A &= \frac{1}{3} \end{aligned}$$

Therefore, our particular solution is

$$y_p(t) = \frac{1}{3}e^t$$

Then our general solution is

$$y_g(t) = y_h(t) + y_p(t) = ce^{-2t} + \frac{1}{3}e^t$$

2. This equation is a bit more complicated. For this reason, we use the integrating factor. First, make sure the coefficient in front of y' is 1. Then our integrating factor is

$$\mu(x) = e^{\int \frac{2}{x}\,dx} = e^{2\ln|x|} = e^{\ln|x^2|} = x^2$$

Multiplying this to both sides of our equation, we get

$$x^2y' + 2xy = 2xe^{x^2}$$

The reason for doing this is that now the left side is the derivative of y times the integrating factor, so we have

$$\frac{d}{dx}\left(yx^2\right) = 2xe^{x^2}$$

Integrating both sides gives us

$$\begin{aligned} yx^2 &= \int 2xe^{x^2}\,dx \\ yx^2 &= e^{x^2} + c \\ y &= \frac{e^{x^2}}{x^2} + \frac{c}{x^2} \end{aligned}$$

3. We use the integrating factor again. In this case, our integrating factor is the following.

$$\mu(x) = e^{\int \tan x\,dx} = e^{\ln|\sec x|} = \sec x$$

Multiply this to both sides and simplify

$$\begin{aligned} y' \sec x + y \sec x \tan x &= 1 \\ \frac{d}{dx}\left(y \sec x\right) &= 1 \end{aligned}$$

Now integrate.

$$\begin{aligned} y \sec x &= \int 1\,dx \\ y \sec x &= x + c \\ y &= x \cos x + c \cos x \end{aligned}$$

Solve the following the initial value problem.

$$\frac{dy}{dx} = x - y,\ y(0) = 2$$

Solution

First, rewrite the equation.

$$\frac{dy}{dx} + y = x$$

Now find the homogeneous solution by solving the characteristic equation.

$$r + 1 = 0$$

$r = -1$ is the only solution, so our homogeneous solution is

$$y = ce^{-t}$$

Then for our particular solution, we guess

$$y_p(t) = Ax + B,\ y_p'(t) = A$$

Plugging this into our equation, we get

$$\begin{aligned} (A) + (Ax + B) &= x \\ Ax + (A + B) &= x \end{aligned}$$

Since we want the right side to be exactly equal to the left side, we must have $A = 1$ and $A + B = 0$. So our solution is $A = 1$ and $B = -1$. Then our particular solution is

$$y_p(t) = x - 1$$

Then our general solution is

$$y_g(t) = y_h(t) + y_p(t) = ce^{-t} + x - 1$$

Now plug in $y(0) = 2$ to solve for c.

$$\begin{aligned} &2 = ce^0 + 0 - 1 \\ &2 = c - 1 \\ &c = 3 \end{aligned}$$

Then our specific solution is

$$y(t) = 3e^{-t} + x - 1$$

3.2 Approximating Solutions

Let $y(x)$ be a solution to the differential equation

$$y' = x^2 y$$

with initial condition $y(1) = 1$. Approximate $y(1.5)$ using Euler's Method with $\Delta x = .1$

Solution

We use the following recurrences.

$$x_{n+1} = x_n + \Delta x, \;\; y_{n+1} = y_n + \Delta x\, y'(x_n, y_n)$$

In our case, this amounts to the following.

$$x_{n+1} = x_n + .1, \;\; y_{n+1} = y_n + .1\, x_n^2\, y_n$$

We set up a table as follows.

n	x_n	y_n
0	1	1
1	-	-
2	-	-
3	-	-
4	-	-
5	-	-

The first column is easy to fill in, since we add .1 each time.

n	x_n	y_n
0	1	1
1	1.1	-
2	1.2	-
3	1.3	-
4	1.4	-
5	1.5	-

(Note: we used this many rows since our final row has $x_n = 1.5$, and we want to estimate y when $x = 1.5$.) Now we use our formula for y_{n+1} to fill in our next column.

n	x_n	y_n
0	1	1
1	1.1	1.1
2	1.2	1.2233
3	1.3	1.3795
4	1.4	1.5858
5	1.5	1.8765

Therefore, we have $y(1.5) \approx 1.8765$.

3.3 Autonomous Differential Equations and Stability

In a confined population, $y(t)$ is the number of people who have contracted a contagious but nonfatal disease after t days. Then $y(t)$ is modeled by the autonomous differential equation

$$y' = ky(P - y)$$

1. Determine the equilibrium solutions to the differential equation
2. Assess the stability of each equilibrium solution
3. Assuming that only one or two people have the disease initially, what happens to the population in the long run?

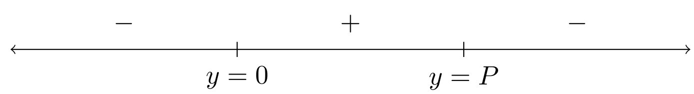
Solution

1. Set the right side of the equation equal to zero and solve for y.

$$ky(P - y) = 0$$

   This has solutions of $y(t) = 0$ and $y(t) = P$. These are our equilibrium solutions. (Note: Equilibrium solutions are specifically constant solutions!)

2. For this, we make a number line and look at signs. For $y < 0$, $y' < 0$, for $0 < y < P$, $y' > 0$, and for $y > P$, $y' < 0$. So we make our sign chart as follows.



   From this, the equilibrium solution $y(t) = 0$ is unstable and the equilibrium solution $y(t) = P$ is stable.

3. From the chart above, we see that if $0 < y < P$, then $y' > 0$, so the population grows. Since the equilibrium solution $y(t) = P$ is stable, the population grows to $y = P$.

4 Higher Order and Systems of Differential Equations

4.1 Higher Order Homogeneous Differential Equations

Solve for the general solution of the given homogeneous differential equations.

1. $y''' + y'' - 6y' = 0$
2. $y'' + 4y' + 4y = 0$
3. $y'' + 2y' + 2y = 0$

Solution

1. Solve the characteristic equation.

$$r^3 + r^2 - 6r = 0$$

First factor out an r to get a quadratic equation. This equation only has solutions of $r = 0$, $r = 2$, and $r = -3$. Then our general solution is the following.

$$y_g(t) = c_1 e^{0t} + c_2 e^{2t} + c_3 e^{-3t}$$

(Note: The first term can simply be written as c_1 since $e^{0t} = 1$.)

2. Solve the characteristic equation.

$$r^2 + 4r + 4 = 0$$

This is a perfect square, so it only has the solution $r = -2$. Then since we have a repeated root, our general solution is the following.

$$y_g(t) = c_1 e^{-2t} + c_2 t e^{-2t}$$

(Note: Any time there is a repeated root, you must put in an extra t in the second term.)

3. Solve the characteristic equation.

$$r^2 + 2r + 2 = 0$$

Using the quadratic formula, the roots are $r = 1 \pm i$. When there are complex roots, the real part remains in the exponential term, while the complex part turns into sines and cosines.

$$y_g(t) = c_1 e^t \sin t + c_2 e^t \cos t$$

(Note: In general, if $r = \alpha \pm i\beta$, we have the following.)

$$y_g(t) = c_1 e^{\alpha t} \sin \beta t + c_2 e^{\alpha t} \cos \beta t$$

4.2 Higher Order Non-homogeneous Differential Equations

Find the solution to the given differential equation

$$y'' + 4y = 16x$$

with initial conditions $y(0) = 2$ and $y'(0) = -3$

Solution

First, find the homogeneous solution by solving the characteristic equation.

$$r^2 + 4 = 0$$

This has solutions of $r = \pm 2i$. So our homogeneous solution is

$$y_h(t) = c_1 \sin 2t + c_2 \cos 2t$$

Now we take care of our particular solution. We make the guess $y_p(t) = Ax + B$. Taking derivatives and plugging them into our equation, we get

$$\begin{aligned} (0) + 4(Ax + B) &= 16x \\ 4Ax + 4B &= 16x \end{aligned}$$

Since we want the left side to be exactly the same as the right, we need $4A = 16$ and $4B = 0$. Therefore, we have $A = 4$ and $B = 0$. Then our particular solution is

$$y_p(t) = 4x$$

Then our general solution is

$$y_g(t) = y_h(t) + y_p(t) = c_1 \sin 2t + c_2 \cos 2t + 4x$$

Now we plug in our initial conditions. Plugging in $y(0) = 2$, we get

$$2 = c_1(0) + c_2(1) + 4(0) = c_2$$

Now take the derivative to plug in $y'(0) = -3$.

$$\begin{aligned} y_g'(t) &= 2c_1 \cos 2t - 2c_2 \sin 2t + 4 \\ -3 &= 2c_1(1) - 2c_2(0) + 4 = 2c_1 + 4 \end{aligned}$$

From these equations, we get $c_1 = -7/2$ and $c_2 = 2$. Then our specific solution is the following.

$$y(t) = -\frac{7}{2} \sin 2t + 2 \cos 2t + 4x$$

4.3 Reduction Methods for Systems of Differential Equations

Use the reduction method to find the general solution of the systems of differential equations

$$x' = y + 2t + 3$$
$$y' = -x + 4t - 2$$

Solution

Take the derivative of the first equation to get the following.

$$x'' = y' + 2$$

Plug the second equation into this to get another equation.

$$x'' = (-x + 4t - 2) + 2$$
$$x'' + x = 4t$$

So now we have a second order differential equation. Solve the homogeneous solution to get

$$x_h(t) = c_1 \sin t + c_2 \cos t$$

Guess $x_p(t) = At + B$ to get $A = 4$ and $B = 0$ so our solution for $x(t)$ is the following.

$$x(t) = c_1 \sin t + c_2 \cos t + 4t$$

Now use solve for y in the first equation to get

$$y(t) = x' - 2t - 3$$

Find $x'(t)$ and plug it in to get

$$y(t) = c_1 \cos t - c_2 \sin t - 2t + 1$$

Then the general solution is the following.

$$x(t) = c_1 \sin t + c_2 \cos t + 4t$$
$$y(t) = c_1 \cos t - c_2 \sin t - 2t + 1$$

4.4 Matrix Methods for Systems of Differential Equations

Use matrix methods to find the general solution and the stability of the origin for the system of differential equations

$$\begin{aligned} x' &= -x + y \\ y' &= 2y \end{aligned}$$

Solution

First, write the system of differential equations in matrix form.

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Now compute the eigenvalues of the matrix.

$$\det \begin{pmatrix} -1-\lambda & 1 \\ 0 & 2-\lambda \end{pmatrix} = (-1-\lambda)(2-\lambda) = 0$$

This has solutions of $\lambda_1 = -1$ and $\lambda_2 = 2$. Then compute the eigenvectors for these corresponding eigenvalues.

$$\begin{pmatrix} -1-\lambda_1 & 1 \\ 0 & 2-\lambda_1 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

From this, we get $\beta = 0$. This is the only condition, so we choose $\alpha = 1$ to get

$$\vec{v}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Similarly, for λ_2, we get

$$\vec{v}_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Then our general solution (in matrix form) is the following.

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{-t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 e^{2t} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} c_1 e^{-t} + c_2 e^{2t} \\ 3c_2 e^{2t} \end{pmatrix}$$

Then pulling out our equations, we get our final answer.

$$\begin{aligned} x(t) &= c_1 e^{-t} + c_2 e^{2t} \\ y(t) &= 3c_2 e^{2t} \end{aligned}$$

Since one eigenvalue is positive and the other is negative, the origin is semi-stable.

4.5 Special Cases in Matrix Methods for Systems of Differential Equations

Find the general solution to the given systems of differential equations.

1. $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

2. $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

Solution

1. Find the eigenvalues as before.

$$\det \begin{pmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{pmatrix} = (2-\lambda)^2 = 0$$

This has only the solution $\lambda = 2$. Since there is only the one solution, this is a special case. The general solution is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{\lambda t} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + te^{\lambda t}\vec{v},\ \vec{v} = (A - \lambda I) \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

Calculating $\vec{v}$, we get

$$\vec{v} = \begin{pmatrix} 2-2 & 1 \\ 0 & 2-2 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} y_0 \\ 0 \end{pmatrix}$$

So we may immediately write

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{2t} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + te^{2t} \begin{pmatrix} y_0 \\ 0 \end{pmatrix}$$

Then from this, we have our solution.

$$x(t) = x_0 e^{2t} + y_0 t e^{2t}$$
$$y(t) = y_0 e^{2t}$$

2. Find the eigenvalues as before.

$$\det \begin{pmatrix} 1-\lambda & -1 \\ 2 & 3-\lambda \end{pmatrix} = \lambda^2 - 4\lambda + 5 = 0$$

Using the quadratic formula, we get that $\lambda = 2 \pm i$. This is a special case since our roots are complex. In this special case, we need only one eigenvalue and one eigenvector. We choose $\lambda = 2 + i$. Now calculate the eigenvector as normal.

$$\begin{pmatrix} 1-\lambda & -1 \\ 2 & 3-\lambda \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} -1-i & -1 \\ 2 & 1-i \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

We choose the vector

$$\vec{v}_1 = \begin{pmatrix} -1 \\ 1+i \end{pmatrix}$$

Then we compute a solution as normal

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{(2+i)t} \begin{pmatrix} -1 \\ 1+i \end{pmatrix}$$

Now we must split this into its real and imaginary parts with some algebra.

$$e^{(2+i)t}\begin{pmatrix}-1\\1+i\end{pmatrix} = e^{2t}(\cos t + i\sin t)\begin{pmatrix}-1\\1+i\end{pmatrix} = \begin{pmatrix}e^{2t}(\cos t + i\sin t)(-1)\\e^{2t}(\cos t + i\sin t)(1+i)\end{pmatrix}$$
$$= \begin{pmatrix}-e^{2t}\cos t - ie^{2t}\sin t\\e^{2t}\cos t + ie^{2t}\cos t + ie^{2t}\sin t - e^{2t}\sin t\end{pmatrix}$$
$$= \begin{pmatrix}-e^{2t}\cos t\\e^{2t}\cos t - e^{2t}\sin t\end{pmatrix} + i\begin{pmatrix}-e^{2t}\sin t\\e^{2t}\cos t + e^{2t}\sin t\end{pmatrix}$$

Then our general solution is a constant times the real part plus a constant times the imaginary part.

$$\begin{pmatrix}x(t)\\y(t)\end{pmatrix} = c_1\begin{pmatrix}-e^{2t}\cos t\\e^{2t}\cos t - e^{2t}\sin t\end{pmatrix} + c_2\begin{pmatrix}-e^{2t}\sin t\\e^{2t}\cos t + e^{2t}\sin t\end{pmatrix}$$

Since the real part of our eigenvalue is positive, the origin is unstable.

4.6 Applications of Differential Equations

On a wildlife preserve, the population of gorillas is modeled by the differential equation

$$\frac{dy}{dt} = .1y(100 - y)$$

Where t is measured in years.

1. Find the equilibrium solutions to the differential equation
2. Compute the general solution to the differential equation
3. Given that the preserve had 45 gorillas 4 years ago, find the number of gorillas currently on the preserve
4. Describe what happens to the gorilla population in the long run

Solution

1. Setting the right side equal to zero and solving for y, we get $y(t) = 0$ and $y(t) = 100$. These are our equilibrium solutions.
2. This equation is separable, so we first separate each variable.

$$\begin{aligned} \frac{dy}{dt} &= .1y(100 - y) \\ \frac{dy}{y(100 - y)} &= .1\,dt \end{aligned}$$

Now integrate both sides.

$$\begin{aligned} \int \frac{dy}{y(100 - y)} &= \int .1\,dt \\ \frac{1}{100}\ln|y| - \frac{1}{100}\ln|100 - y| &= .1t + C \\ \ln|y| - \ln|100 - y| &= 10t + C \\ \ln\left|\frac{y}{100 - y}\right| &= 10t + C \\ \frac{y}{100 - y} &= Ce^{10t} \\ \frac{100 - y}{y} &= Ce^{-10t} \\ \frac{100}{y} - 1 &= Ce^{-10t} \\ \frac{100}{y} &= 1 + Ce^{-10t} \\ y &= \frac{100}{1 + Ce^{-10t}} \end{aligned}$$

(Note: We skipped the partial fractions in the first integral. The algebra may be done differently, but the same result can be obtained.)

3. The data in the problem tells us that $y(0) = 45$ and we would like to calculate $y(4)$. Plugging our initial data into the equation from the previous part, we have the following.

$$45 = \frac{100}{1 + C}$$

Solving for C gives us that $C = 11/9$. Then our equation for the number of gorillas is

$$y(t) = \frac{100}{1 + 11/9\, e^{-10t}}$$

Plugging in $t = 4$ gives us $y(4) = 100$.

4. The equilibrium solutions are $y(t) = 0$ and $y(t) = 100$. Assessing the stability of these solutions shows that $y(t) = 100$ is an stable solution, so if there are any gorillas on the reserve, the population will tend towards 100 gorillas in the long run.