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1 Limits

Evaluate the following limits.

1. $\displaystyle\lim_{x\to 3} \frac{2x-6}{x^2-9}$
2. $\displaystyle\lim_{x\to 0^+} \frac{x+x^2}{\sqrt{x}}$
3. $\displaystyle\lim_{x\to 0} \frac{\sin(6x)}{x}$
4. $\displaystyle\lim_{x\to 4} \frac{x^2-16}{x^2+4}$

Solution

1. Begin by factoring and canceling the common factor.

$$\lim_{x\to 3} \frac{2x-6}{x^2-9} = \lim_{x\to 3} \frac{2(x-3)}{(x-3)(x+3)} = \lim_{x\to 3} \frac{2}{x+3} = \frac{2}{6} = \frac{1}{3}$$

2. Split it into two limits and solve each of them.

$$\lim_{x\to\ 0} \frac{x+x^2}{\sqrt{x}} = \lim_{x\to 0} \frac{x}{\sqrt{x}} + \lim_{x\to 0} \frac{x^2}{\sqrt{x}} = \lim_{x\to 0} \sqrt{x} + \lim_{x\to 0} x^{3/2} = 0 + 0 = 0$$

3. Use the identity

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$$

as follows. (multiply the top and bottom by 6)

$$\lim_{x\to 0} \frac{\sin(6x)}{x} = \lim_{x\to 0} \frac{6\sin(6x)}{6x} = 6 \lim_{6x\to 0} \frac{\sin(6x)}{6x} = 6(1) = 6$$

4. Note that since the function is continuous and the denominator is nonzero, we may plug in 4 directly.

$$\lim_{x\to 4} \frac{x^2-16}{x^2+4} = \frac{0}{20} = 0$$

2 Derivatives

2.1 Difference Quotients

For the following function, (a) formulate the difference quotient in terms of x and h, (b) simplify the difference quotient, (c) take the limit as $h \to 0$ to find the derivative.

$$f(x) = \frac{1}{\sqrt{x}}$$

Solution

(a) Write out the formula for the difference quotient.

$$\frac{f(x+h) - f(x)}{h} = \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h}$$

(b) Now simplify the expression above.

$$\frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} = \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} \frac{\sqrt{x+h}\sqrt{x}}{\sqrt{x+h}\sqrt{x}} = \frac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x+h}\sqrt{x}}$$

This seems a bit messy, but we now multiply by the conjugate of the numerator to simplify further.

$$\frac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x+h}\sqrt{x}} = \frac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x+h}\sqrt{x}} \frac{\sqrt{x} + \sqrt{x+h}}{\sqrt{x} + \sqrt{x+h}} = \frac{-h}{h\sqrt{x+h}\sqrt{x}(\sqrt{x} + \sqrt{x+h})}$$

Now we can cancel the h term from the top and bottom.

$$\frac{-h}{h\sqrt{x+h}\sqrt{x}(\sqrt{x} + \sqrt{x+h})} = \frac{-1}{\sqrt{x+h}\sqrt{x}(\sqrt{x} + \sqrt{x+h})}$$

(c) Now take the limit. Notice that since the denominator is not zero at $h = 0$, we can plug in $h = 0$ directly here.

$$\lim_{h \to 0} \frac{-1}{\sqrt{x+h}\sqrt{x}(\sqrt{x} + \sqrt{x+h})} = \frac{-1}{\sqrt{x}^2(2\sqrt{x})} = \frac{-1}{2x^{3/2}}$$

2.2 Average Rate of Change

Find the average rate of change of the function

$$f(x) = \sqrt{x+2}$$

as x changes from 2 to 7.

Solution

The average rate of change from $x = 2$ to $x = 7$ is defined to be

$$\frac{f(7) - f(2)}{7 - 2} = \frac{\sqrt{9} - \sqrt{4}}{7 - 2} = \frac{1}{5}.$$

2.3 Derivative Rules

Evaluate the derivatives of the following functions.

1. $f(x) = \dfrac{3}{x} + \dfrac{x}{3}$
2. $f(x) = 3(x^2 + x)^3$
3. $y = \ln\left(\dfrac{x^2}{1+x^2}\right)$
4. $y = e^{\sin x}$
5. $q(v) = \cos(\ln v)$

Solution

1. Use the power rule .

$$f'(x) = \frac{-3}{x^2} + \frac{1}{3}$$

2. Use the power rule and chain rule.

$$f'(x) = 9(x^2 + 3)^2(2x)$$

3. First, simplify with logarithm rules.

$$y = \ln\left(\frac{x^2}{1+x^2}\right) = \ln x^2 - \ln(1+x^2) = 2\ln x - \ln(1+x^2)$$

Now take the simplified derivative.

$$y'(x) = \frac{2}{x} - \frac{1}{1+x^2}(2x)$$

4. Use the chain rule.

$$y' = e^{\sin x}(\cos x)$$

5. Use the chain rule.

$$q'(v) = -\sin(\ln v)\left(\frac{1}{v}\right)$$

2.4 Tangent Lines

Find the equation of the tangent line to the function

$$f(x) = (x-1)e^x + 1$$

at $x = 1$.

Solution

To find the equation of the tangent line, we need a point and a slope. The point in question is at $x = 1$. Since $f(1) = 1$, the point we need is $(1, 1)$. To find the slope of the tangent line, take the derivative.

$$f'(x) = (1)e^x + (x-1)e^x = xe^x$$

At $x = 1$, $f'(1) = e$. Therefore, the slope of the tangent line at $x = 1$ is e. We may then write our equation (in point-slope form) as follows.

$$y - 1 = e(x-1) \quad \text{or} \quad y = ex - e + 1$$

2.5 Curve Sketching

Consider the function

$$y = x^3 - 9x^2 + 24x - 10$$

(a) Find the critical points of this function and determine if these points are points of relative maxima or relative minima. (b) Find the intervals where the function is increasing or decreasing. (c) Find the inflection points of this function. (d) Find the intervals where the graph of this function is concave up and concave down. (e) Sketch the graph of this function.

Solution

(a) To find the critical points, we first find the derivative and set it equal to zero.

$$y' = 3x^2 - 18x + 24 = 0 \implies x = 2,\ x = 4$$

The points $(2, 10)$ and $(4, 6)$ are the only critical points. To determine whether these are relative maxima or relative minima, we use the second derivative test.

$$y'' = 6x - 18$$

At $x = 2$, we see $y''(2) = -6 < 0$ so $(2, 10)$ is a relative maxima. At $x = 4$, $y''(4) = 6 > 0$ so $(4, 6)$ is a relative minima.

(b) To find intervals where y is increasing or decreasing, we use a number line for y' and test between our critical points.

$$+ \quad x = 2 \quad - \quad x = 4 \quad + \qquad y'$$

From this, we see that y is increasing for $x < 2$ and $x > 4$ (in interval notation, $(-\infty, 2) \cup (4, \infty)$) and decreasing for $2 < x < 4$ (in interval notation, $(2, 4)$).

(c) To find the inflection points, we look at when the second derivative is zero.

$$y'' = 6x - 18 = 0 \implies x = 3$$

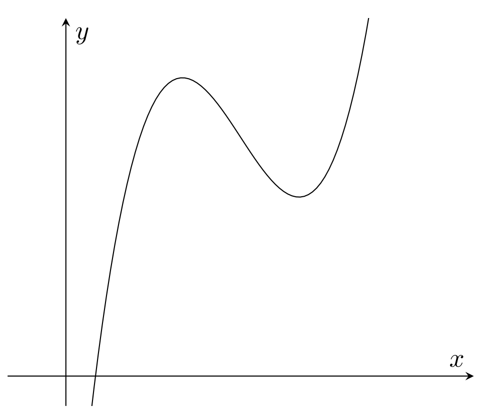
To show this is an inflection point, look at a number line for y''.

$$- \quad x = 3 \quad + \qquad y''$$

Since the sign of y'' changes at $x = 3$, $(3, 8)$ is an inflection point (and the only one).

(d) From the number line from the previous part, we see that $y'' < 0$ for $x < 3$ and $y'' > 0$ for $x > 3$. This implies that the graph of y is concave down for $x < 3$ and concave up for $x > 3$.

(e) The graph of the function looks as follows.



Consider the function

$$y = \frac{x+1}{x+4}$$

(a) Find the critical points of this function and determine if these points are points of relative maxima or relative minima. (b) Find the intervals where the function is increasing or decreasing. (c) Find the inflection points of this function. (d) Find the intervals where the graph of this function is concave up and concave down. (e) Sketch the graph of this function.

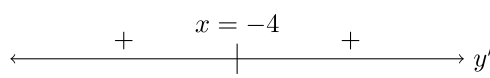
Solution

(a) To find the critical points, we find the derivative and find where it is equal to zero (or is underfined).

$$y' = \frac{(x+4)(1) - (x+1)(1)}{(x+4)^2} = \frac{3}{(x+4)^2}$$

This function is never zero, but it is undefined if $x = -4$. Therefore, we have a critical value at $x = -4$. This function does not have any critical points (no point at $x = -4$) so it does not have any relative maxima or relative minima.

(b) To find where the function is increasing or decreasing, we use a number line for y'.



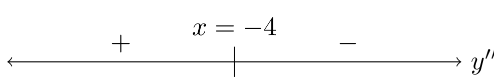
From this, we see that the function is increasing for $x \neq -4$ (in interval notation $(-\infty, -4) \cup (-4, \infty)$).

(c) To find the inflection point(s), we find the zeros (and critical points) of the second derivative.

$$y'' = \frac{-6}{(x+4)^3}$$

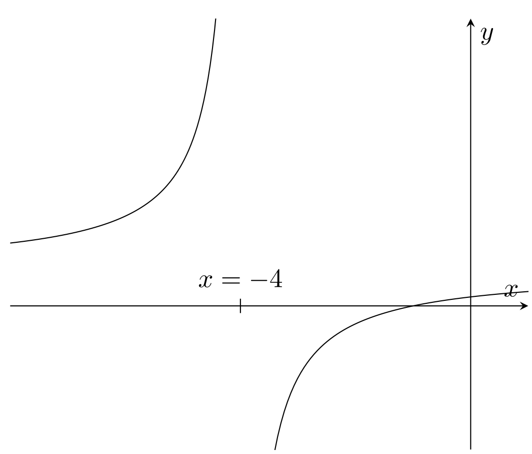
The second derivative is never zero, but has a critical value at $x = -4$. Therefore, the function has no inflection point.

(d) To find where the graph of y is concave up or concave down, we make a number line for y''.



From this, we can say that the graph of y is concave up for $x > -4$ (in interval notation $(-4, \infty)$) and concave down for $x < -4$ (in interval notation $(-\infty, -4)$).

(e) The graph of the curve looks like the following.



3 Applications of Derivatives

3.1 Instantaneous Rates of Change

Given a ball whose position, in feet, above the ground is given by $y = 50t - 16t^2$, where t is the time, find

(a) how far off the ground is the ball when $t = 1$.

(b) the average velocity from $t = 0$ to $t = 2$.

(c) the instantaneous velocity at $t = 1$.

Solution

(a) To find the height of the ball at $t = 1$, plug in $t = 1$.

$$y(1) = 34ft$$

(b) The average velocity of the ball from $t = 0$ to $t = 2$ is defined as

$$\frac{y(2) - y(0)}{2 - 0} = \frac{36 - 0}{2 - 0} = 18$$

(c) The instantaneous velocity of the ball at $t = 1$ is defined as the derivative of the height function at $t = 1$. Therefore, the instantaneous velocity at $t = 1$ is given by the following.

$$y'(1) = 50 - 32(1) = 18ft/s$$

3.2 Optimization

Find two positive numbers whose product is 36 and whose sum is a minimum.

Solution

Let x and y be numbers such that $xy = 36$. We would like to minimize their sum $S = x + y$. Using the fact that $y = 36/x$, we can rewrite S as $S = x + 36/x$. We would like to minimize this function, so we take the derivative and set it equal to zero.

$$S' = 1 - \frac{36}{x^2} = 0 \implies x = \pm 6$$

Since x and y are positive, we have that $x = 6$ and $y = 36/x = 36/6 = 6$. To show this is a minimum, we use the second derivative test.

$$S'' = \frac{72}{x^3} \implies S''(6) = \frac{72}{216} > 0$$

Since $S''(6) > 0$, $x = 6$, $y = 6$ is a minimum as desired.

A drug company is designing an open-top rectangular box with a square base, that will hold 32000 cubic centimeters. Determine the dimensions x and y that will yield the minimum surface area.

Solution

We would like an open-top, rectangular box with a square base of length l, width w, and height h of volume 32,000cm^3. Since the base is square, we know that $l = w$. Since the volume must be fixed at 32,000cm^3, we have the following relationship.

$$l^2h = 32,000cm^3$$

(using that $l = w$) Now we would like to minimize the function

$$SA = lw + 2lh + 2wh = l^2 + 4lh.$$

Using the relationship from above, we notice the following.

$$l^2h = 32,000cm^3 \implies h = 32,000/l^2 \implies SA = l^2 + \frac{128,000}{l}$$

Now we take the derivative and set it equal to zero to minimize our function.

$$(SA)' = 2l - \frac{128,000}{l^2} = 0 \implies l = \sqrt[3]{64,000}cm$$

Use the second derivative test to show that this is actually a minimum value.

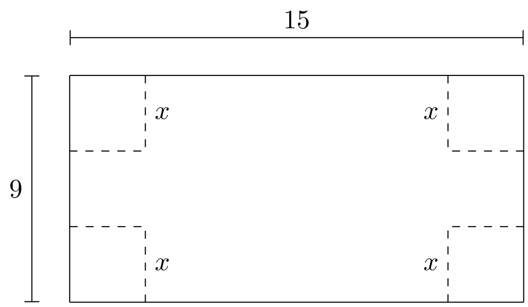
$$(SA)'' = 2 + \frac{256,000}{l^3} \implies (SA)''(\sqrt[3]{64,000}) = 2 + 4 = 6 > 0$$

Since $(SA)'' > 0$, $l = \sqrt[3]{64,000}cm$ is a minimum. The size of our box is then $l = \sqrt[3]{64,000}cm$ and $w = \frac{1}{2}\sqrt[3]{64,000}cm$.

An open-top box is to be constructed from a 9×15 inch piece of cardboard by cutting out the same size square from each of the four corners and then folding up the resulting sides. Find the largest possible volume of such a box.

Solution

First, draw the picture as follows.



From this picture, we see the length of the box will be $15 - 2x$, the width of the box will be $9 - 2x$ and the height of the box will be x. Notice that we must have $0 \le x \le 4.5$ (otherwise we cut out too much!). We now need a formula for the volume of the box (depending on x).

$$V = (15 - 2x)(8 - 2x)(x) = 4x^3 - 48x^2 + 120x$$

To maximize this, we take the derivative and set it equal to zero.

$$V' = 12x^2 - 96x + 120 = 0 \implies x = 4 \pm \sqrt{6}$$

Notice that $4 + \sqrt{6} > 4.5$, so this is outside of our possible x values. We look only at $4 - \sqrt{6}$. To see whether this is a relative maximum, we use the second derivative test.

$$V'' = 24x - 96 \implies V''(4 - \sqrt{6}) = -24\sqrt{6} < 0$$

Since $V''(4 - \sqrt{6}) < 0$, $x = 4 - \sqrt{6}$ gives the maximum value.

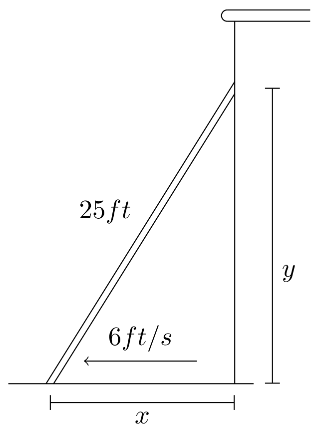
(Note: Here, we should have worried about checking the volume at the endpoints, $x = 0$ and $x = 4.5$. In these cases however, the volume of the box is zero and is definitely not our maximum volume!)

3.3 Related Rates

A ladder 25 ft long leans against a vertical wall. If the lower end is being moved away from the wall at a rate of 6 ft/sec, how fast is the height of the top decreasing when the lower end is 7 ft from the wall?

Solution

Draw the picture as follows.



With the picture above, the problem is to compute dy/dt when $x = 7ft$. From the picture, we see that we must always have the relationship below.

$$x^2 + y^2 = 25^2$$

Now we take the derivative with respect to time to find

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \implies \frac{dy}{dt} = \frac{-x}{y}\frac{dx}{dt}.$$

When $x = 7$, we see that $y = \sqrt{25^2 - 7^2} = \sqrt{574}$. Using the fact that $dx/dt = 6ft/s$, we get the result that

$$\frac{dy}{dt} = \frac{-7}{\sqrt{574}}(6)ft/s = \frac{-42}{\sqrt{574}}ft/s.$$

Which means that the height of the ladder is falling at a rate of $\frac{42}{\sqrt{574}}ft/s$.

4 Integrals

4.1 Indefinite Integrals

Evaluate the following integrals.

1. $\int x(x-2)^3\,dx$
2. $\int \frac{x+x^2}{\sqrt{x}}\,dx$
3. $\int \tan x\,dx$
4. $\int \frac{dx}{x\ln x}$
5. $\int xe^{-x}\,dx$

Solution

1. First, expand the integrand.

$$\int x(x-6)^3\,dx = \int x\left(x^3-6x^2+12x-8\right)\,dx = \int \left(x^4-6x^3+12x^2-8x\right)\,dx$$

Now integrate term by term using the power rule.

$$\int \left(x^4-6x^3+12x^2-8x\right)\,dx = \frac{x^5}{5}-\frac{3x^4}{2}+4x^3-4x^2+C$$

2. Separate the integrand.

$$\int \frac{x+x^2}{\sqrt{x}}\,dx = \int \left(\frac{x}{\sqrt{x}}+\frac{x^2}{\sqrt{x}}\right)\,dx = \int \left(x^{1/2}+x^{3/2}\right)\,dx$$

Now integrate term by term using power rule again.

$$\int \left(x^{1/2}+x^{3/2}\right)\,dx = \frac{2}{3}x^{3/2}+\frac{2}{5}x^{5/2}+C$$

3. Rewrite $\tan x$ as follows.

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$$

Now use the substitution $u=\cos x$. We see $du=-\sin x\,dx$. This turns our integral into the following.

$$\int \frac{\sin x}{\cos x}\,dx = \int -\frac{du}{u}$$

Now integrate and back substitute.

$$\int -\frac{du}{u} = -\ln|u|+C = -\ln|\cos x|+C$$

4. Begin with the substitution $u = \ln x$. We see $du = dx/x$ and our integral becomes the following.

$$\int \frac{dx}{x \ln x} = \int \frac{du}{u}$$

Now integrate and back substitute.

$$\int \frac{du}{u} = \ln |u| + C = \ln |\ln x| + C$$

5. This problem will require the use of integration by parts. Make the following choice of u and dv.

$$\begin{aligned} u &= x & dv &= e^{-x}\,dx \\ du &= dx & v &= -e^{-x} \end{aligned}$$

Using the formula for integration by parts, we then see the following.

$$\int xe^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x} + C$$

4.2 Definite Integrals

Evaluate the following definite integrals.

1. $\displaystyle\int_1^3 5x \ln x \, dx$
2. $\displaystyle\int_1^2 e^{2x} \, dx$
3. $\displaystyle\int_0^{\pi} \sin x \, dx$

Solution

1. This problem requires integration by parts. Make the following choice of u and dv.

$$\begin{aligned} u &= \ln x & dv &= 5x\,dx \\ du &= dx/x & v &= 5x^2/2 \end{aligned}$$

Our integral then becomes the following.

$$\int_1^3 5x \ln x \, dx = \frac{5}{2}x^2 \ln x\Big|_1^3 - \int_1^3 \frac{5}{2}x \, dx = \left(\frac{5}{2}x^2 \ln x - \frac{5}{4}x^2\right)\Big|_1^3 = \left(\frac{45}{2}\ln 3 - \frac{45}{4}\right) - \left(0 - \frac{5}{4}\right) = \frac{45}{2}\ln 3 - 10$$

2. Make use of the substitution $u = 2x$. Then $du = 2dx$ and our integral becomes the following.

$$\int_1^2 e^{2x} \, dx = \int_2^4 e^u \frac{1}{2} \, du = \frac{1}{2}e^u\Big|_2^4 = \frac{1}{2}(e^4 - e^2)$$

3. Integrate directly.

$$\int_0^{\pi} \sin x \, dx = -\cos x\Big|_0^{\pi} = (1) - (-1) = 2$$

4.3 Areas between Functions

Find the area bounded by $y = \sin x$ and $y = \cos x$ when $0 \leq x \leq \pi/4$.

Solution

All we need to set up our integral is to determine which curve is the top curve and which curve is the bottom curve. A graph of $\sin x$ and $\cos x$ shows that $\cos x \geq \sin x$ for $0 \leq x \leq \pi/4$. Therefore, the top curve is $\cos x$ and the bottom curve is $\sin x$. We then set up our integral as follows.

$$\int_0^{\pi/4} (\cos x - \sin x)\, dx$$

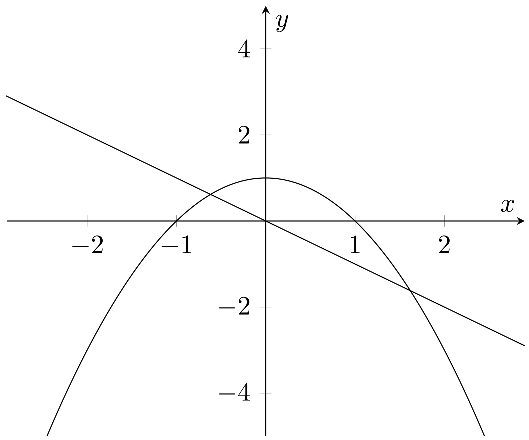
Integrate directly from here.

$$\int_0^{\pi/4} (\cos x - \sin x)\, dx = (\sin x + \cos x)\Big|_0^{\pi/4} = \left(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right) - (0 + 1) = \sqrt{2} - 1$$

Set up but do not evaluate the integral that expresses the area bounded by $y = 1 - x^2$ and $y = -x$.

Solution

First, draw a graph to get an idea of what area we are finding.



Now we find the two points of intersection by setting the curves equal.

$$1 - x^2 = -x \implies x^2 - x - 1 = 0 \implies x = \frac{1 \pm \sqrt{5}}{2}$$

From the graph above, we see that between these two points (where the area is enclosed), $y_1 = 1 - x^2$ is the top curve and $y_2 = -x$ is the bottom curve. We then set up our integral as follows.

$$\int_{\frac{1-\sqrt{5}}{2}}^{\frac{1+\sqrt{5}}{2}} \Big((1 - x^2) - (-x)\Big)\, dx$$

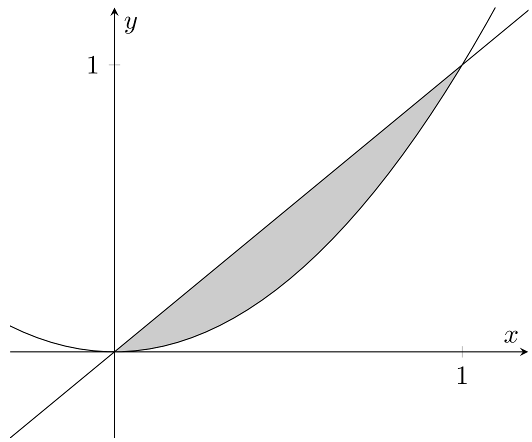
4.4 Volumes of Revolution

Look at the region bounded by $y = x$ and $y = x^2$ from $x = 0$ to $x = 1$.

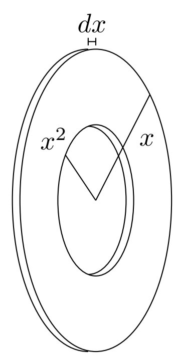
1. Calculate the volume generated from rotating this region about the x-axis.
2. Calculate the volume generated from rotating this region about the y-axis.

Solution

1. First, we graph the area in question.



We choose to use the disk method here. A picture of a disk is shown below.



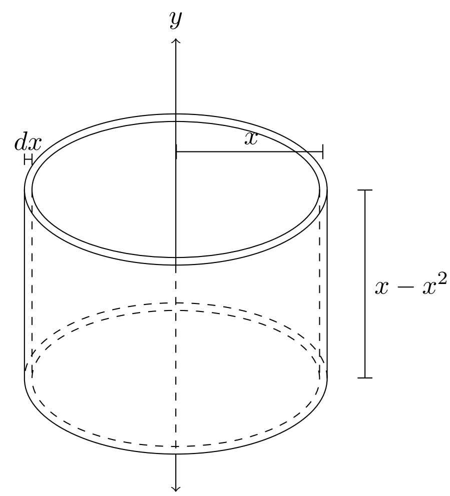
The volume of each of these is then

$$dV = \pi(x)^2\,dx - \pi(x^2)^2\,dx = \pi((x)^2 - (x^2)^2)\,dx.$$

Integrating these volumes from $x = 0$ to $x = 1$, we get the result that

$$V = \int_0^1 \pi((x)^2 - (x^2)^2)\,dx = \pi\int_0^1 (x^2 - x^4)\,dx = \pi\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{2\pi}{15}$$

2. To rotate about the y-axis, we use the shell method. Using the same graph as above, we draw our shell as follows.



From this picture, the volume of each shell is given by

$$dV = 2\pi x(x - x^2)\, dx.$$

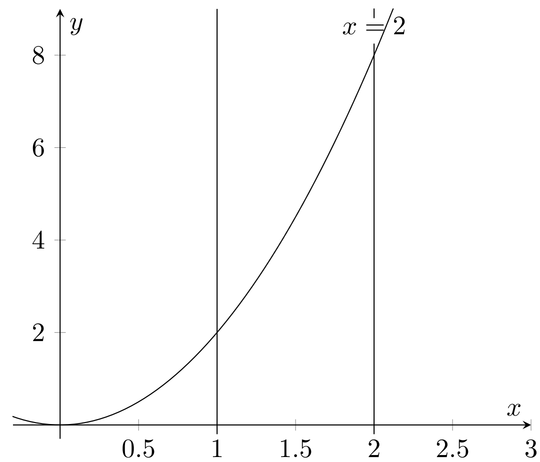
Integrating these from $x = 0$ to $x = 1$ gives us

$$V = \int_0^1 2\pi x(x - x^2)\, dx = 2\pi \int_0^1 \left(x^2 - x^3\right)\, dx = 2\pi \left(\frac{1}{3} - \frac{1}{4}\right) = \frac{\pi}{6}.$$

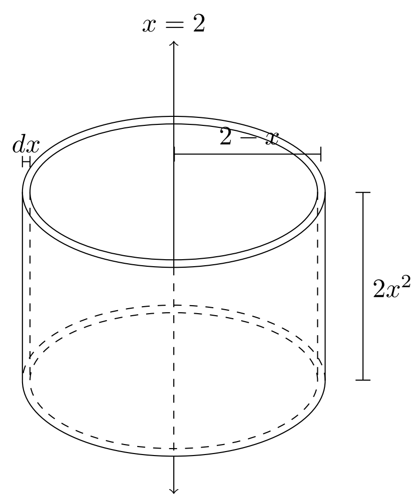
Find the volume of the solid generated by revolving the region bounded by $y = 2x^2$, $x = 1$, and the x-axis about the line $x = 2$.

Solution

As before, we first draw the graph.



We choose to use the shell method (discs requires two separate integrals). A typical shell looks like the following.



A typical shell then has volume

$$dV = 2\pi(2-x)(2x^2)\,dx.$$

We then get the result that

$$V = \int_1^2 2\pi(2-x)(2x^2)\,dx = 2\pi\int_1^2 (4x^2 - 2x^3)\,dx = 2\pi\left(\frac{28}{3} - \frac{15}{2}\right) = \frac{11\pi}{3}$$

4.5 Improper Integrals

Determine which of the following improper integrals are convergent. Evaluate those that are convergent.

1. $\displaystyle\int_e^\infty \frac{\ln(x)}{x}\,dx$
2. $\displaystyle\int_0^\infty 2xe^{-x^2}\,dx$

Solution

1. Write the infinite bound as a limit.

$$\int_e^\infty \frac{\ln x}{x}\,dx = \lim_{t\to\infty}\int_e^t \frac{\ln x}{x}\,dx$$

   Now integrate (use the substitution $u = \ln x$).

$$\lim_{t\to\infty}\int_e^t \frac{\ln x}{x}\,dx = \lim_{t\to\infty}\left(\frac{1}{2}\ln^2 t - \frac{1}{2}\right)$$

   As $t \to \infty$, this limit does not exist, so the integral is divergent.

2. As before, write the integral as a limit.

$$\int_0^\infty 2xe^{-x^2}\,dx = \lim_{t\to\infty}\int_0^t 2xe^{-x^2}\,dx$$

   Now integrate (use the substitution $u = -x^2$).

$$\lim_{t\to\infty}\int_0^t 2xe^{-x^2}\,dx = \lim_{t\to\infty}\left(-e^{-t^2}+1\right)$$

   As $t \to \infty$, $e^{-t^2} \to 0$ implies that we have

$$\lim_{t\to\infty}\left(-e^{-t^2}+1\right) = 0 + 1 = 1.$$

   Therefore, the integral is convergent and its value is 1.