# Contents

# 1 Differentiation

## 1.1 Review from Math 22100

Find the derivative of each of the following functions

1. $y = 3\ln\sqrt[3]{t^2+2}$
2. $y = \ln\dfrac{x}{2x-1}$
3. $y = \dfrac{e^x}{x^2}$

## Solution

Use logarithm rules to simplify the first two. Use quotient rule on the third.

1. $y' = \dfrac{2t}{t^2+2}$
2. $y' = \dfrac{1}{x} - \dfrac{2}{2x-1}$
3. $y' = \dfrac{e^x(x-2)}{x^3}$

### 1.2 L'Hospital's Rule

Evaluate the following limits using L'Hospital's rule

1. $\lim_{x \to 0} \dfrac{1 - e^x}{2x}$
2. $\lim_{x \to 0} \dfrac{x - \sin x}{x}$

## Solution

Both are already in the form $\frac{0}{0}$, so apply L'Hospital's rule directly.

1. $\lim_{x \to 0} \dfrac{1 - e^x}{2x} \overset{H}{=} \lim_{x \to 0} \dfrac{-e^x}{2} = \dfrac{-1}{2}$
2. $\lim_{x \to 0} \dfrac{x - \sin x}{x} \overset{H}{=} \lim_{x \to 0} \dfrac{1 - \cos x}{1} = 0$

### 1.3 Applications of Derivatives

Find the minima and maxima, the points of inflection, and sketch the graph of the curve below.

$$y = xe^{-x}$$

## Solution

To calculate the extremum, find the zeros of the first derivative.

$$y'(x) = (x-1)e^{-x} = 0$$

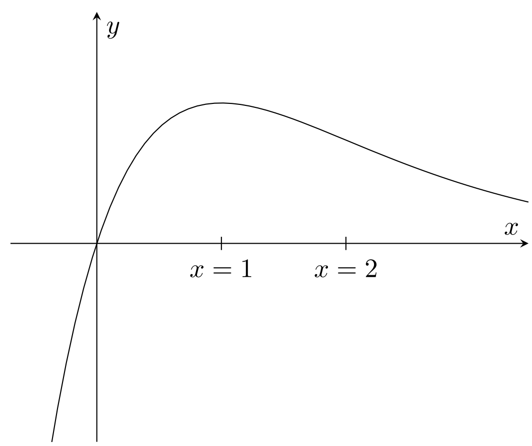
Since $e^{-x} > 0$, we only have the solution $x = 1$. Therefore, the only possible relative extremum is at $x = 1$. To determine whether this is a maximum of minimum, use the second derivative test.

$$y''(1) = -e^{-1} < 0$$

Therefore, $(1, e^{-1})$ is a local maximum.
To determine any points of inflection, find the zeros of the second derivative.

$$y''(x) = (x-2)e^{-x} = 0$$

Since $e^{-x} > 0$, we only have the solution $x = 2$. Since the second derivatives changes sign here (check values less than 2 and greater than 2), this value is an inflection point.
To sketch the curve, it helps to find any asymptotes and where the function is increasing/decreasing. In doing so, we see the curve has a horizontal asymptote of $y = 0$, increasing on $(-\infty, 1)$, and decreasing on $(1, \infty)$. We can then see our graph has the following shape.



## 1.4 Newton's Method

Find (approximately) a positive root of the following equation.

$$4\sin x - x = 0$$

(Note: Not all classes cover this material.)

## Solution

Make a table of values using the relation

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Where $f(x) = 4\sin x - x$. We may choose (almost) any value for $x_0$. We choose $x_0 = 2$. The table is the following

| $n$ | $x_n$ |
|---|---|
| 0 | 2 |
| 1 | 2.6144 |
| 2 | 2.4793 |
| 3 | 2.4746 |
| 4 | 2.4746 |

The values remain the same (up to 4 decimal points) after $n = 4$. Therefore, $x = 2.4746$ is an approximate solution to our equation.

# 2 Integration

## 2.1 General Power Rule Integrals

Using the general power rule to evaluate the following integrals

1. $\int e^{2x}\sqrt{1+e^{2x}}\,dx$

2. $\int (1-\cos 5x)^3 \sin 5x\,dx$

3. $\int_1^e \frac{\sqrt{\ln x}}{x}\,dx$

## Solution

1. Use the substitution $u = 1 + e^{2x}$.

$$\int e^{2x}\sqrt{1+e^{2x}}\,dx = \int \frac{1}{2}\sqrt{u}\,du = \frac{1}{3}u^{3/2} + c = \frac{1}{3}(1+e^{2x})^{3/2} + c$$

2. Use the subsitution $u = 1 - \cos 5x$.

$$\int (1-\cos 5x)^3 \sin 5x\,dx = \int \frac{1}{5}u^3\,du = \frac{1}{20}u^4 + c = \frac{1}{20}(1-\cos 5x)^4 + c$$

3. Use the substitution $u = \ln x$.

$$\int_1^e \frac{\sqrt{\ln x}}{x}\,dx = \int_0^1 \sqrt{u}\,du = \frac{2}{3}u^{3/2}\Big|_0^1 = \frac{2}{3}$$

### 2.2 Logarithmic and Exponential Integrals

Evaluate the following integrals

1. $\int te^{t^2}\,dx$
2. $\int 2^x\,dx$
3. $\int \frac{1}{x\ln x}\,dx$

## Solution

1. Use the substitution $u = t^2$.

$$\int te^{t^2}\,dt = \int \frac{1}{2}e^u\,du = \frac{1}{2}e^u + c = \frac{1}{2}e^{t^2} + c$$

2. Use the formula for integration of exponential functions.

$$\int 2^x\,dx = \frac{1}{\ln 2}2^x + c$$

3. Use the substitution $u = \ln x$.

$$\int \frac{1}{x\ln x}\,dx = \int \frac{1}{u}\,du = \ln|u| + c = \ln|\ln x| + c$$

## 2.3 Trigonometric Integrals

Evaluate the following integrals

1. $\int x^2 \sec x^3 \tan x^3 \, dx$

2. $\int x^3 \sec x^4 \, dx$

## Solution

1. Use the substitution $u = x^3$.

$$\int x^2 \sec x^3 \tan x^3 \, dx = \int \frac{1}{3} \sec u \tan u \, du = \frac{1}{3} \sec u + c = \frac{1}{3} \sec x^3 + c$$

2. Use the substitution $u = x^4$.

$$\int x^3 \sec x^4 \, dx = \int \frac{1}{4} \sec u \, du = \frac{1}{4} \ln |\sec u + \tan u| + c = \frac{1}{4} \ln |\sec x^4 + \tan x^4| + c$$

Evaluate the following integrals

1. $\int \sin^5 x \cos^6 x \, dx$
2. $\int \sin^2 x \cos^2 x \, dx$

## Solution

1. Separate one of the $\sin x$ terms and write the rest in terms of $\cos x$.

$$\int \sin^5 x \cos^6 x \, dx = \int (\sin^2 x)^2 \cos^6 x \sin x \, dx = \int (1 - \cos^2 x)^2 \cos^6 x \sin x \, dx$$

Now use the substitution $u = \cos x$.

$$\begin{aligned}\int (1 - \cos^2 x)^2 \cos^6 x \sin x \, dx &= \int -(1-u^2)^2 u^6 \, du = \int \left(-u^6 + 2u^8 - u^{10}\right) \, du \\ &= \frac{-1}{7}u^7 + \frac{2}{9}u^9 - \frac{1}{11}u^{11} + c \\ &= \frac{-1}{7}\cos^7 x + \frac{2}{9}\cos^9 x - \frac{1}{11}\cos^1 1x + c\end{aligned}$$

2. Use the trigonometric identities

$$\sin 2\theta = 2 \sin\theta \cos\theta, \;\; \sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$$

to simplify as follows.

$$\int \sin^2 x \cos^2 x \, dx = \int (\sin x \cos x)^2 \, dx = \int \frac{1}{4} \sin^2 2x \, dx = \int \frac{1}{8}(1 - \cos 4x) \, dx$$

Now integrate as normal.

$$\int \left(\frac{1}{8} - \frac{1}{8}\cos 4x\right) \, dx = \frac{1}{8}x - \frac{1}{32}\sin 4x + c$$

### 2.4 Inverse Trigonometric Forms

Use trigonometric substitution to evaluate the following integral

$$\int \frac{1}{\sqrt{5-3x^2}}\,dx$$

## Solution

First, simplify the problem.

$$\int \frac{1}{\sqrt{5-3x^2}}\,dx = \frac{1}{\sqrt{3}}\int \frac{1}{\sqrt{5/3-x^2}}\,dx$$

Now use the substitution $x = \sqrt{5/3}\sin\theta$ (since $a = \sqrt{5/3}$).

$$\begin{aligned}\frac{1}{\sqrt{3}}\int \frac{1}{\sqrt{5/3-x^2}}\,dx &= \frac{1}{\sqrt{3}}\int \frac{1}{\sqrt{5/3\cos^2\theta}}\sqrt{\frac{5}{3}}\cos\theta\,d\theta \\ &= \frac{1}{\sqrt{3}}\int d\theta = \frac{1}{\sqrt{3}}\theta + c = \frac{1}{\sqrt{3}}\sin^{-1}\left(\frac{x}{\sqrt{5/3}}\right) + c\end{aligned}$$

(Note: In the last step, to back substitute, we use the fact that $x = \sqrt{5}3\sin\theta$, then solved for $\sin\theta$ and took the inverse.)

### 2.5 Trigonometric Substitution

Use a trigonometric substitution to evaluate the following integral.

$$\int \frac{\sqrt{x^2-9}}{x}\,dx$$

## Solution

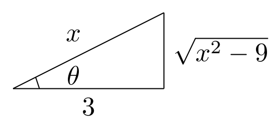
Use the substitution $x = 3\sec\theta$.

$$\int \frac{\sqrt{x^2-9}}{x}\,dx = \int \frac{\sqrt{9\tan^2\theta}}{3\sec\theta} 3\sec\theta\tan\theta\,d\theta = \int 3\tan^2\theta\,d\theta$$

Now use the identity $\tan^2\theta = \sec^2\theta - 1$.

$$\int 3\tan^2\theta\,d\theta = \int 3\sec^2\theta - 3\,d\theta = 3\tan\theta - 3\theta + c$$

Now make a triangle to back substitute.




This gives us

$$\tan\theta = \frac{1}{3}\sqrt{x^2-9},\ \theta = \sec^{-1}\left(\frac{x}{3}\right)$$

Therefore, we have

$$\int \frac{\sqrt{x^2-9}}{x}\,dx = \sqrt{x^2-9} - 3\sec^{-1}\left(\frac{x}{3}\right) + c$$

### 2.6 Integration by Parts

Utilize integration by parts to evaluate the following integrals

1. $\int xe^{-x}\,dx$
2. $\int \cot^{-1} x\,dx$

## Solution

1. Choose $u = x$ and $dv = e^{-x}\,dx$ to obtain the following.

$$\int xe^{-x}\,dx = -xe^{-x} - \int -e^{-x}\,dx = -xe^{-x} - e^{-x} + c$$

2. Choose $u = \cot^{-1} x$ and $dv = dx$ to obtain the following.

$$\int \cot^{-1} x\,dx = x\cot^{-1} x - \int \frac{-x}{x^2+1}\,dx$$

Now apply a $u$-substitution to evaluate the new integral to get the following.

$$\int \cot^{-1} x\,dx = x\cot^{-1} x + \frac{1}{2}\ln|x^2+1| + c$$

## 2.7 Integration of Rational Functions

Use either long division or partial fractions to evaluate the following integral.

$$\int \frac{x^3+3x}{(x^2+1)^2}\,dx$$

## Solution

Begin by applying partial fractions.

$$\frac{x^3+3x}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$$

Multiplying both sides by $(x^2+1)^2$ gives

$$\begin{aligned} x^3+3x &= (Ax+B)(x^2+1)+(Cx+D) \\ x^3+3x &= Ax^3+Bx^2+(A+C)x+(B+D) \end{aligned}$$

Since we would like the left side to be exactly the same as the right side, we must have the following.

$$\begin{aligned} A &= 1 \\ B &= 0 \\ A+C &= 3 \\ B+D &= 0 \end{aligned}$$

Which has the solution of

$$A=1,\ B=0, C=2, D=0$$

Therefore, we have the following.

$$\frac{x^3+3x}{(x^2+1)^2} = \frac{x}{x^2+1} + \frac{2x}{(x^2+1)^2}$$

So we have

$$\int \frac{x^3+3x}{(x^2+1)^2}\,dx = \int \frac{x}{x^2+1} + \frac{2x}{(x^2+1)^2}\,dx = \frac{1}{2}\ln|x^2+1| - \frac{1}{x^2+1} + c$$

(Note: We skipped the integration of the two functions. They can be done using $u$-substitutions.)

# 3 Series

## 3.1 Geometric Series

Find the sum of the following series.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^{n-1}}$$

## Solution

First, rewrite the series to look a bit nicer.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^{n-1}} = \sum_{n=1}^{\infty} \left(\frac{-1}{2}\right)^{n-1}$$

From this, we have $r = -1/2$. Since $|r| < 1$, the series converges. Then using the formula

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$$

We have the following.

$$\sum_{n=1}^{\infty} \left(\frac{-1}{2}\right)^{n-1} = \frac{1}{1-(-1/2)} = \frac{2}{3}$$

### 3.2 Tests for Convergence

Determine whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{n}{n^2+1}$$

(Note: Not all classes cover this material.)

## Solution

Use limit comparison test and compare $n/(n^2+1)$ to $1/n$.

$$\lim_{n\to\infty} \frac{\frac{n}{n^2+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^2}} = 1 \neq 0$$

Therefore, the series in the problem converges if $\sum \frac{1}{n}$ converges and diverges if $\sum \frac{1}{n}$ diverges. Since the series $\sum \frac{1}{n}$ diverges, the series of the problem diverges as well.

### 3.3 Maclaurin Series

Find the first three nonzero terms in the Maclaurin series for the following functions

1. $y = \cos x$
2. $y = \ln(1 + x)$

## Solution

1. Use the formula for the MacLaurin series. The first few derivatives are as follows.

$$f(0) = 1,\ f'(0) = 0,\ f''(0) = -1,\ f'''(0) = 0,\ f''''(0) = 1$$

So we have the following.

$$\cos x \approx 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$$

2. Use the formula for the MacLaurin series. The first few derivatives are as follows.

$$f(0) = 0,\ f'(0) = 1,\ f''(0) = -1,\ f'''(0) = 2$$

So we have the following.

$$\ln(1 + x) \approx x - \frac{1}{2}x^2 + \frac{1}{3}x^3$$

### 3.4 Operations with Series

Find the Maclaurin series of the following function.

$$y = \ln(1 + x^2)$$

## Solution 1

The following series should be familiar to us already.

$$\ln(1 + z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n}{n}$$

If we let $z = x^2$, we arrive at the following result.

$$\ln(1 + x^2) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n}}{n}$$

## Solution 2

Differentiating the function gives us the following.

$$y' = \frac{2x}{1 + x^2} = \frac{2x}{1 - (-x^2)}$$

Using the formula

$$\frac{a}{1 - r} = \sum_{n=1}^{\infty} ar^{n-1}$$

we have

$$y' = \frac{2x}{1 - (-x^2)} = \sum_{n=1}^{\infty} 2x \left(-x^2\right)^{n-1} = \sum_{n=1}^{\infty} 2(-1)^{n-1} x^{2n-1}$$

To get back to our function $y$, we now integrate.

$$y = \int y' \, dx = \int \sum_{n=1}^{\infty} 2(-1)^{n-1} x^{2n-1} \, dx = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n}}{n} + c$$

Plugging in $x = 0$ shows us that

$$y(0) = \ln(1 + 0^2) = 0, \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 0^{2n}}{n} + c = 0 + c = c$$

So we must have $c = 0$. Therefore, we have

$$\ln(1 + x^2) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n}}{n}$$

### 3.5 Computations with Series

Use the first 4 terms of the Maclaurin series for $y = e^{-x}$ to approximate the value of $e^{-0.2}$. Determine the error of your approximation.

## Solution

The first four terms of the MacLaurin series for $y = e^{-x}$ is the following.

$$y = e^{-x} \approx 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3$$

Plugging in $x = 0.2$ gives us

$$y(-0.2) = e^{-0.2} \approx 1 - .2 + \frac{1}{2}(.2)^2 - \frac{1}{6}(.2)^3 = .81867$$

The exact value of $e^{-0.2}$ is approximately .81873. Therefore, the error given by our estimate is approximately .00006.

## 3.6 Fourier Series

Determine the Fourier series for the following function on the given interval

$$f(t) = \begin{cases} 0 \text{ if } -1 < t \leq 0 \\ t \text{ if } 0 < t < 1 \end{cases}$$

## Solution

First, determine the $a_0$ term.

$$a_0 = \frac{1}{1}\int_{-1}^{1} f(t)\,dt = \int_0^1 t\,dt = \frac{1}{2}t^2\Big|_0^1 = \frac{1}{2}$$

Now determine the $a_n$ terms.

$$a_n = \frac{1}{1}\int_{-1}^{1} f(t)\cos(nt)\,dt = \int_0^1 t\cos(nt)\,dt = \frac{1}{n}t\sin(nt) + \frac{1}{n^2}\cos(nt)\Big|_0^1 = \frac{\sin(n)}{n} + \frac{\cos(n)}{n^2} - \frac{1}{n^2}$$

Similarly, determine the $b_n$ terms.

$$b_n = \frac{1}{1}\int_{-1}^{1} f(t)\sin(nt) = \int_0^1 t\sin(nt)\,dt = \frac{-1}{n}t\cos(nt) + \frac{1}{n^2}\sin(nt)\Big|_0^1 = \frac{-\cos(n)}{n} + \frac{\sin(n)}{n^2} + \frac{1}{n}$$

Then the Fourier series is the following.

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nt) + b_n\sin(nt)$$

Where $a_n$ and $b_n$ are the values found above.

# 4 First-Order Differential Equations

## 4.1 Solutions to Differential Equations

Show that the function

$$y = xe^{-2x} + 3e^{-2x}$$

is a solution to the given differential equation.

$$\frac{dy}{dx} + 2y = e^{-2x}$$

## Solution

To show that the function given is a solution, we first compute its derivative.

$$y = xe^{-2x} + 3e^{-2x},\ y' = -2xe^{-2x} - 5e^{-2x}$$

Now we plug this values ($y$ and $y'$) in our differential equation for $y$ and its derivative.

$$\left(-2xe^{-2x} - 5e^{-2x}\right) + 2\left(xe^{-2x} + 3e^{-2x}\right) = -2xe^{-2x} - 5e^{-2x} + +2xe^{-2x} + 6e^{-2x} = e^{-2x}$$

Therefore, our function satisfies the differential equation. This means that the function is a solution.

### 4.2 Separation of Variables

Find the general solution to the given differential equations

1. $dx + (2\cos^2 x - y\cos^2 x)\, dy = 0$
2. $xe^y\, dx + e^{-x}\, dy = 0$

## Solution

1. First, simplify the equation.

$$\begin{aligned} dx + (2\cos^2 x - y\cos^2 x)\, dy &= 0 \\ dx + \cos^2 x(2-y)\, dy &= 0 \\ \sec^2 x\, dx + (2-y)\, dy &= 0 \end{aligned}$$

Now that the variables are separated (there are no terms with both $x$ and $y$) we can integrate.

$$\begin{aligned} \int \sec^2 x\, dx + \int (2-y)\, dy &= c \\ \tan x + 2y - \frac{1}{2}y^2 &= c \end{aligned}$$

This is the general solution to the differential equation.

2. Simplify the equation as before.

$$\begin{aligned} xe^y\, dx + e^{-x}\, dy &= 0 \\ xe^x\, dx + e^{-y}\, dy &= 0 \end{aligned}$$

(Divide by $e^{-x}$ and then by $e^y$) Now integrate.

$$\begin{aligned} \int xe^x\, dx + \int e^{-y}\, dy &= c \\ (x-1)e^x - e^{-y} &= c \end{aligned}$$

(We skipped the integration by parts in the first integral) This is the general solution to the differential equation.

### 4.3 First-Order Linear Differential Equations

Find the solution to the following differential equation.

$$2\frac{dy}{dx} - 8xy = e^{2x^2}$$

## Solution

First, we need the equation to have no term in front of $\frac{dy}{dx}$, so we simplify.

$$\begin{aligned} 2\frac{dy}{dx} - 8xy &= e^{2x^2} \\ \frac{dy}{dx} - 4xy &= \frac{1}{2}e^{2x^2} \end{aligned}$$

Now calculate the integrating factor.

$$\mu(x) = e^{\int -4x\,dx} = e^{-2x^2}$$

Now multiply this to both sides of the equation.

$$e^{-2x^2}\frac{dy}{dx} - 4xe^{-2x^2}y = \frac{1}{2}$$

The reason for this is that now the left side of the equation can be simplified to the derivative of $y$ times our integrating factor.

$$\begin{aligned} e^{-2x^2}\frac{dy}{dx} - 4xe^{-2x^2}y &= \frac{1}{2} \\ \frac{d}{dx}\left(ye^{-2x^2}\right) &= \frac{1}{2} \end{aligned}$$

Now integrate both sides and simplify.

$$\begin{aligned} \int \frac{d}{dx}\left(ye^{-2x^2}\right)dx &= \int \frac{1}{2}\,dx \\ ye^{-2x^2} &= \frac{1}{2}x + c \\ y &= \frac{1}{2}xe^{2x^2} + ce^{2x^2} \end{aligned}$$

This is the general solution to the differential equation.

### 4.4 Applications of Differential Equations

A bacteria culture is known to increase at a rate proportional to the number of bacteria present. It is observed that the size of the culture triples in 3 hours. After how many hours should it be 10 times as large?

## Solution

The information of the problem tells us that

$$\frac{dP}{dt} = kP$$

Where $P$ is the number of bacteria. Let the initial amount of bacteria be $P_0$ (at $t = 0$). Separating variables and integrating shows that we have

$$P(t) = Ce^{kt}$$

Now plug in $t = 0$ to get

$$P(0) = P_0,\ P(0) = Ce^0 = C$$

So we have

$$P(t) = P_0 e^{kt}$$

The problem states that when $t = 3$, $P = 3P_0$ (triples in 3 hours). So we have

$$P(3) = 3P_0 = P_0 e^{3k}$$

Solving for $k$ shows that

$$k = \frac{1}{3}\ln 3$$

So we have

$$P(t) = P_0 e^{\frac{1}{3}t\ln 3}$$

The problem now is to determine for what value of $t$ gives $P(t) = 10P_0$. Setting $P(t) = 10P_0$ gives us

$$\begin{aligned} 10P_0 &= P_0 e^{\frac{1}{3}t\ln 3} \\ 10 &= e^{\frac{1}{3}t\ln 3} \\ \ln 10 &= \frac{1}{3}t\ln 3 \\ \frac{3\ln 10}{\ln 3} &= t \end{aligned}$$

# 5 Higher Order Differential Equations

## 5.1 Higher-Order Homogeneous Differential Equations

Find the general solution to the given differential equations

1. $6\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$
2. $2D^2y - 3Dy + y = 0$

## Solution

1. Set up and solve the auxiliary equation.

$$\begin{aligned} 6r^2 - r - 2 &= 0 \\ (2r+1)(3r-2) &= 0 \\ r = \frac{-1}{2},\ r &= \frac{2}{3} \end{aligned}$$

   Then our general solution is the following.

$$y(t) = c_1 e^{-\frac{1}{2}t} + c_2 e^{\frac{2}{3}t}$$

2. Set up and solve the auxiliary equation as before.

$$\begin{aligned} 2r^2 - 3r + 1 &= 0 \\ (2r-1)(r-1) &= 0 \\ r = \frac{1}{2},\ r &= 1 \end{aligned}$$

   Then our general solution is the following.

$$y(t) = c_1 e^{\frac{1}{2}t} + c_2 e^{t}$$

### 5.2 Auxiliary Equations

Solve the following differential equations.

1. $(D^2 + 25)y = 0$
2. $(D^2 - 3D + 5)y = 0$

## Solution

1. The auxiliary equation is

$$r^2 + 25 = 0$$

Which has roots $r = \pm 5i$. Then since these values are complex, the general solution is

$$y(t) = c_1 \sin(5t) + c_2 \cos(5t)$$

2. The auxiliary equation is

$$r^2 - 3r + 5 = 0$$
$$r = \frac{3 \pm \sqrt{9 - 4(1)(5)}}{2} = \frac{3 \pm i\sqrt{11}}{2}$$

Since these values are complex, the imaginary parts turn into sines and cosines as follows.

$$y(x) = c_1 e^{\frac{3}{2}x} \sin\left(\frac{\sqrt{11}}{2}x\right) + c_2 e^{\frac{3}{2}x} \cos\left(\frac{\sqrt{11}}{2}x\right)$$

### 5.3 Non-homogeneous Differential Equations

Find the general solution to the given differential equations.

$$(D^2 - D + 2)y = 4e^{3x}$$

## Solution

First solve the auxiliary equation to determine the homogeneous solution.

$$r^2 - r + 2 = 0$$
$$r = \frac{1 \pm \sqrt{1 - 4(1)(2)}}{2} = \frac{1 \pm i\sqrt{7}}{2}$$

So the homogeneous equation is the following.

$$y_h(x) = c_1 e^{\frac{1}{2}x} \sin\left(\frac{\sqrt{7}}{2}x\right) + c_2 e^{\frac{1}{2}x} \cos\left(\frac{\sqrt{7}}{2}x\right)$$

To determine the particular solution, we guess that it has the form $y_p(t) = Ae^{3x}$. Then we have

$$y_p(x) = Ae^{3x}$$
$$Dy_p(x) = 3Ae^{3x}$$
$$D^2 y_p(x) = 9Ae^{3x}$$

Plugging these into our equation, we get the following.

$$9Ae^{3x} - 3Ae^{3x} + 2Ae^{3x} = 4e^{3x}$$
$$8Ae^{3x} = 4e^{3x}$$
$$A = 2$$

Therefore, our particular equation is $y_p(t) = 2e^{3x}$. Our general solution has the form $y_g = y_h + y_p$. So we have

$$y_g(t) = c_1 e^{\frac{1}{2}x} \sin\left(\frac{\sqrt{7}}{2}x\right) + c_2 e^{\frac{1}{2}x} \cos\left(\frac{\sqrt{7}}{2}x\right) + 2e^{3x}$$

### 5.4 Applications of Second-Order Equations

A 2 lb weight stretches a spring 6 in. The weight is pushed 7 in above the equilibrium position and released. Find the motion of the weight as a function of time, assuming no damping.

## Solution

Using the equations

$$F = kx,\ F = mg$$

and using the information of the problem, we have if $x = 0$, the force of the spring is equal to the force of gravity, so we can see

$$6k = 2(386.09),\ k = 128.697$$

Then using the equation for spring position, we have

$$2D^2y + 128.697y = 0$$

(Where $y$ is distance from equilibrium) Solving this gives us

$$y(t) = c_1 \sin(64.35t) + c_2 \cos(64.35t)$$

Our initial conditions are when $t = 0$, $y = 7$, $Dy = 0$. Plugging these in, we arrive at

$$y(t) = 7\cos(64.35t)$$

### 5.5 Computing the Laplace Transformation

Verify the identity.

$$L\{\sin at\} = \frac{a}{s^2 + a^2}$$

## Solution

Begin with the definition of Laplace transformation.

$$L\{\sin ax\} = \int_0^\infty e^{-sx} \sin ax \, dx = \lim_{t\to\infty} \int_0^t e^{-sx} \sin ax \, dx$$

This integral takes a bit of work to solve. Integration by parts is the usual way of doing it, and it will be omitted.

$$\begin{aligned} \lim_{t\to\infty} \int_0^t e^{-sx} \sin ax \, dx &= \lim_{t\to\infty} \left. \frac{-ae^{-sx} \cos ax - se^{-sx} \sin ax}{s^2 + a^2} \right|_0^t \\ &= \lim_{t\to\infty} \frac{-ae^{-st} \cos at - se^{-st} \sin at}{s^2 + a^2} + \frac{a}{s^2 + a^2} \\ &= \frac{a}{s^2 + a^2} \end{aligned}$$

(You may need to use L'Hospital's rule to evaluate the limit!)

### 5.6 Computing the Inverse Laplace Transformation

Compute the inverse Laplace transformation of the function.

$$F(s) = \frac{5s}{s^2 + 6}$$

## Solution

We know the following.

$$L^{-1}\left(\frac{a}{s^2 + a^2}\right) = \sin at, \ L^{-1}\left(\frac{s}{s^2 + a^2}\right) = \cos at$$

So we would like to rewrite what we have in terms of these formulas. Since there is an $s$ term in the numerator, we will use the second equation. Note that the 5 constant does not affect the inverse Laplace, so we have

$$L^{-1}\left(\frac{5s}{s^2 + 6}\right) = 5L^{-1}\left(\frac{s}{s^2 + (\sqrt{6})^2}\right) = 5\cos\left(\sqrt{6}t\right)$$

Compute the inverse Laplace transformation of the function.

$$F(s) = \frac{s}{(s-1)(s+3)}$$

## Solution

We will first do partial fractions to make the problem simpler.

$$\frac{s}{(s-1)(s+3)} = \frac{1/4}{s-1} + \frac{3/4}{s+3}$$

Then use the rule

$$L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$$

To obtain

$$L^{-1}\left(\frac{s}{(s-1)(s+3)}\right) = \frac{1}{4}L^{-1}\left(\frac{1}{s-1}\right) + \frac{3}{4}L^{-1}\left(\frac{1}{s+3}\right) = \frac{1}{4}e^{t} + \frac{3}{4}e^{-3t}$$

### 5.7 Solving Differential Equations Using Laplace Transformations

Use Laplace transformations to solve the following differential equation

$$y'' - 4y' + 4y = e^{3t},\ y(0) = 0,\ y'(0) = -2$$

## Solution

First, apply the Laplace transformation to both sides of the equation and simplify.

$$\begin{aligned} L(y'') - 4L(y') + 4L(y) &= L(e^{3t}) \\ \left(s^2L(y) - sy(0) - y'(0)\right) - 4\left(sL(y) - y(0)\right) + 4L(y) &= \frac{1}{s-3} \\ L(y)\left(s^2 - 4s + 4\right) + 2 &= \frac{1}{s-3} \\ L(y)(s-2)^2 &= \frac{1}{s-3} - 2 \\ L(y) &= \frac{1}{(s-3)(s-2)^2} - \frac{2}{(s-2)^2} \end{aligned}$$

Now we must apply partial fractions to the right side to simplify further.

$$\frac{1}{(s-3)(s-2)^2} = \frac{1}{s-3} + \frac{-1}{s-2} + \frac{-1}{(s-2)^2}$$

Then use the rules

$$L^{-1}\left(\frac{1}{s-a}\right) = e^{at},\ L^{-1}\left(\frac{n!}{(s-a)^n}\right) = t^ne^{at}$$

To obtain the final answer.

$$y = L^{-1}\left(\frac{1}{s-3}\right) - L^{-1}\left(\frac{1}{s-2}\right) - 3L^{-1}\left(\frac{1!}{(s-2)^2}\right) = e^{3t} - e^{2t} - 3te^{2t}$$