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1 Differentiation

1.1 Review from Math 22100

Find the derivative of each of the following functions

1. \( y = 3 \ln \sqrt[3]{t^2 + 2} \)
2. \( y = \ln \frac{x}{2x - 1} \)
3. \( y = \frac{e^x}{x^2} \)

Solution

Use logarithm rules to simplify the first two. Use quotient rule on the third.

1. \( y' = \frac{2t}{t^2 + 2} \)
2. \( y' = \frac{1}{x} - \frac{2}{2x - 1} \)
3. \( y' = \frac{e^x(x - 2)}{x^3} \)
1.2 L'Hospital's Rule

Evaluate the following limits using L'Hospital's rule

1. \( \lim_{x \to 0} \frac{1 - e^x}{2x} \)

2. \( \lim_{x \to 0} \frac{x - \sin x}{x} \)

Solution

Both are already in the form \( \frac{0}{0} \), so apply L'Hospital's rule directly.

1. \( \lim_{x \to 0} \frac{1 - e^x}{2x} = \lim_{x \to 0} \frac{-e^x}{2} = \frac{-1}{2} \)

2. \( \lim_{x \to 0} \frac{x - \sin x}{x} = \lim_{x \to 0} \frac{1 - \cos x}{1} = 0 \)
1.3 Applications of Derivatives

Find the minima and maxima, the points of inflection, and sketch the graph of the curve below.

\[ y = xe^{-x} \]

**Solution**

To calculate the extremum, find the zeros of the first derivative.

\[ y'(x) = (x - 1)e^{-x} = 0 \]

Since \( e^{-x} > 0 \), we only have the solution \( x = 1 \). Therefore, the only possible relative extremum is at \( x = 1 \). To determine whether this is a maximum or minimum, use the second derivative test.

\[ y''(1) = -e^{-1} < 0 \]

Therefore, \((1, e^{-1})\) is a local maximum.

To determine any points of inflection, find the zeros of the second derivative.

\[ y''(x) = (x - 2)e^{-x} = 0 \]

Since \( e^{-x} > 0 \), we only have the solution \( x = 2 \). Since the second derivatives changes sign here (check values less than 2 and greater than 2), this value is an inflection point. To sketch the curve, it helps to find any asymptotes and where the function is increasing/decreasing. In doing so, we see the curve has a horizontal asymptote of \( y = 0 \), increasing on \((-\infty, 1)\), and decreasing on \((1, \infty)\). We can then see our graph has the following shape.
1.4 Newton’s Method

Find (approximately) a positive root of the following equation.

\[ 4 \sin x - x = 0 \]

(Note: Not all classes cover this material.)

**Solution**

Make a table of values using the relation

\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \]

Where \( f(x) = 4 \sin x - x \). We may choose (almost) any value for \( x_0 \). We choose \( x_0 = 2 \). The table is the following

<table>
<thead>
<tr>
<th>( n )</th>
<th>( x_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>2.6144</td>
</tr>
<tr>
<td>2</td>
<td>2.4793</td>
</tr>
<tr>
<td>3</td>
<td>2.4746</td>
</tr>
<tr>
<td>4</td>
<td>2.4746</td>
</tr>
</tbody>
</table>

The values remain the same (up to 4 decimal points) after \( n = 4 \). Therefore, \( x = 2.4746 \) is an approximate solution to our equation.
2 Integration

2.1 General Power Rule Integrals

Using the general power rule to evaluate the following integrals

1. \( \int e^{2x} \sqrt{1 + e^{2x}} \, dx \)

2. \( \int (1 - \cos 5x)^3 \sin 5x \, dx \)

3. \( \int_1^e \frac{\sqrt{\ln x}}{x} \, dx \)

Solution

1. Use the substitution \( u = 1 + e^{2x} \).

\[
\int e^{2x} \sqrt{1 + e^{2x}} \, dx = \int \frac{1}{2} \sqrt{u} \, du = \frac{1}{3} u^{3/2} + c = \frac{1}{3} (1 + e^{2x})^{3/2} + c
\]

2. Use the subsitution \( u = 1 - \cos 5x \).

\[
\int (1 - \cos 5x)^3 \sin 5x \, dx = \int \frac{1}{5} u^3 \, du = \frac{1}{20} u^4 + c = \frac{1}{20} (1 - \cos 5x)^4 + c
\]

3. Use the substitution \( u = \ln x \).

\[
\int_1^e \frac{\sqrt{\ln x}}{x} \, dx = \int_0^1 \sqrt{u} \, du = \frac{2}{3} u^{3/2} \bigg|_0^1 = \frac{2}{3}
\]
2.2 Logarithmic and Exponential Integrals

Evaluate the following integrals

1. \( \int te^{t^2} \, dx \)

2. \( \int 2^x \, dx \)

3. \( \int \frac{1}{x \ln x} \, dx \)

Solution

1. Use the substitution \( u = t^2 \).

\[
\int te^{t^2} \, dt = \int \frac{1}{2} e^u \, du = \frac{1}{2} e^u + c = \frac{1}{2} e^{t^2} + c
\]

2. Use the formula for integration of exponential functions.

\[
\int 2^x \, dx = \frac{1}{\ln 2} 2^x + c
\]

3. Use the substitution \( u = \ln x \).

\[
\int \frac{1}{x \ln x} \, dx = \int \frac{1}{u} \, du = \ln |u| + c = \ln |\ln x| + c
\]
2.3 Trigonometric Integrals

Evaluate the following integrals

1. $\int x^2 \sec x^3 \tan x^3 \, dx$

2. $\int x^3 \sec x^4 \, dx$

Solution

1. Use the substitution $u = x^3$.

$$\int x^2 \sec x^3 \tan x^3 \, dx = \int \frac{1}{3} \sec u \tan u \, du = \frac{1}{3} \sec u + c = \frac{1}{3} \sec x^3 + c$$

2. Use the substitution $u = x^4$.

$$\int x^3 \sec x^4 \, dx = \int \frac{1}{4} \sec u \, du = \frac{1}{4} \ln |\sec u + \tan u| + c = \frac{1}{4} \ln |\sec x^4 + \tan x^4| + c$$
Evaluate the following integrals

1. \( \int \sin^5 x \cos^6 x \, dx \)

2. \( \int \sin^2 x \cos^2 x \, dx \)

Solution

1. Separate one of the \( \sin x \) terms and write the rest in terms of \( \cos x \).

\[
\int \sin^5 x \cos^6 x \, dx = \int (\sin^2 x)^2 \cos^6 x \sin x \, dx = \int (1 - \cos^2 x)^2 \cos^6 x \sin x \, dx
\]

Now use the substitution \( u = \cos x \).

\[
\int (1 - \cos^2 x)^2 \cos^6 x \sin x \, dx = \int -(1 - u^2)^2 u^6 \, du = \int \left(-u^6 + 2u^8 - u^{10}\right) \, du
\]

\[
= \frac{-1}{7}u^7 + \frac{2}{9}u^9 - \frac{1}{11}u^{11} + c
\]

\[
= \frac{-1}{7}\cos^7 x + \frac{2}{9}\cos^9 x - \frac{1}{11}\cos^{11} x + c
\]

2. Use the trigonometric identities

\[
\sin 2\theta = 2 \sin \theta \cos \theta, \quad \sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)
\]

to simplify as follows.

\[
\int \sin^2 x \cos^2 x \, dx = \int (\sin x \cos x)^2 \, dx = \int \frac{1}{4} \sin^2 2x \, dx = \int \frac{1}{8}(1 - \cos 4x) \, dx
\]

Now integrate as normal.

\[
\int \left(\frac{1}{8} - \frac{1}{8}\cos 4x\right) \, dx = \frac{1}{8}x - \frac{1}{32}\sin 4x + c
\]
2.4 Inverse Trigonometric Forms

Use trigonometric substitution to evaluate the following integral

\[
\int \frac{1}{\sqrt{5 - 3x^2}} \, dx
\]

Solution

First, simplify the problem.

\[
\int \frac{1}{\sqrt{5 - 3x^2}} \, dx = \frac{1}{\sqrt{3}} \int \frac{1}{\sqrt{\frac{5}{3} - x^2}} \, dx
\]

Now use the substitution \( x = \sqrt{\frac{5}{3}} \sin \theta \) (since \( a = \sqrt{\frac{5}{3}} \)).

\[
\frac{1}{\sqrt{3}} \int \frac{1}{\sqrt{\frac{5}{3} - x^2}} \, dx = \frac{1}{\sqrt{3}} \int \frac{1}{\sqrt{\frac{5}{3} \cos^2 \theta} \sqrt{\frac{5}{3}}} \cos \theta \, d\theta
\]

\[
= \frac{1}{\sqrt{3}} \int d\theta = \frac{1}{\sqrt{3}} \theta + c = \frac{1}{\sqrt{3}} \sin^{-1} \left( \frac{x}{\sqrt{\frac{5}{3}}} \right) + c
\]

(Note: In the last step, to back substitute, we use the fact that \( x = \sqrt{\frac{5}{3}} \sin \theta \), then solved for \( \sin \theta \) and took the inverse.)
2.5 Trigonometric Substitution

Use a trigonometric substitution to evaluate the following integral.

\[
\int \frac{\sqrt{x^2 - 9}}{x} \, dx
\]

Solution

Use the substitution \( x = 3 \sec \theta \).

\[
\int \frac{\sqrt{x^2 - 9}}{x} \, dx = \int \frac{\sqrt{9 \tan^2 \theta}}{3 \sec \theta} 3 \sec \theta \tan \theta \, d\theta = \int 3 \tan^2 \theta \, d\theta
\]

Now use the identity \( \tan^2 \theta = \sec^2 \theta - 1 \).

\[
\int 3 \tan^2 \theta \, d\theta = \int 3 \sec^2 \theta - 3 \, d\theta = 3 \tan \theta - 3\theta + c
\]

Now make a triangle to back substitute.

\[
\begin{tikzpicture}
\draw (0,0) -- (30:3) -- (0,0) ;
\draw (0,0) -- (0,0) node[anchor=west] {3} ;
\draw (30:3) -- (30:0.25) node[anchor=north] {x} ;
\draw (0,0) -- (30:2.5) node[anchor=south] {\sqrt{x^2 - 9}} ;
\end{tikzpicture}
\]

This gives us

\[
\tan \theta = \frac{1}{3} \sqrt{x^2 - 9}, \quad \theta = \sec^{-1} \left( \frac{x}{3} \right)
\]

Therefore, we have

\[
\int \frac{\sqrt{x^2 - 9}}{x} \, dx = \sqrt{x^2 - 9} - 3 \sec^{-1} \left( \frac{x}{3} \right) + c
\]
2.6 Integration by Parts

Utilize integration by parts to evaluate the following integrals

1. \( \int xe^{-x} \, dx \)
2. \( \int \cot^{-1} x \, dx \)

Solution

1. Choose \( u = x \) and \( dv = e^{-x} \, dx \) to obtain the following.

\[
\int xe^{-x} \, dx = -xe^{-x} - \int -e^{-x} \, dx = -xe^{-x} - e^{-x} + c
\]

2. Choose \( u = \cot^{-1} x \) and \( dv = dx \) to obtain the following.

\[
\int \cot^{-1} x \, dx = x \cot^{-1} x - \int \frac{-x}{x^2 + 1} \, dx
\]

Now apply a \( u \)-substitution to evaluate the new integral to get the following.

\[
\int \cot^{-1} x \, dx = x \cot^{-1} x + \frac{1}{2} \ln |x^2 + 1| + c
\]
2.7 Integration of Rational Functions

Use either long division or partial fractions to evaluate the following integral.

\[ \int \frac{x^3 + 3x}{(x^2 + 1)^2} \, dx \]

Solution

Begin by applying partial fractions.

\[
\frac{x^3 + 3x}{(x^2 + 1)^2} = \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{(x^2 + 1)^2}
\]

Multiplying both sides by \((x^2 + 1)^2\) gives

\[
x^3 + 3x = (Ax + B)(x^2 + 1) + (Cx + D)
\]

Since we would like the left side to be exactly the same as the right side, we must have the following.

\[
A = 1 \\
B = 0 \\
A + C = 3 \\
B + D = 0
\]

Which has the solution of

\[
A = 1, \quad B = 0, \quad C = 2, \quad D = 0
\]

Therefore, we have the following.

\[
\frac{x^3 + 3x}{(x^2 + 1)^2} = \frac{x}{x^2 + 1} + \frac{2x}{(x^2 + 1)^2}
\]

So we have

\[
\int \frac{x^3 + 3x}{(x^2 + 1)^2} \, dx = \int \frac{x}{x^2 + 1} + \frac{2x}{(x^2 + 1)^2} \, dx = \frac{1}{2} \ln |x^2 + 1| - \frac{1}{x^2 + 1} + c
\]

(Note: We skipped the integration of the two functions. They can be done using \(u\)-substitutions.)
3 Series

3.1 Geometric Series

Find the sum of the following series.

\[ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^{n-1}} \]

Solution

First, rewrite the series to look a bit nicer.

\[ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2^{n-1}} = \sum_{n=1}^{\infty} \left( \frac{-1}{2} \right)^{n-1} \]

From this, we have \( r = -1/2 \). Since \( |r| < 1 \), the series converges. Then using the formula

\[ \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1 - r} \]

We have the following.

\[ \sum_{n=1}^{\infty} \left( \frac{-1}{2} \right)^{n-1} = \frac{1}{1 - (-1/2)} = \frac{2}{3} \]
3.2 Tests for Convergence

Determine whether the following series converges or diverges.

\[ \sum_{n=1}^{\infty} \frac{n}{n^2 + 1} \]

(Note: Not all classes cover this material.)

Solution

Use limit comparison test and compare \( \frac{n}{n^2 + 1} \) to \( \frac{1}{n} \).

\[ \lim_{n \to \infty} \frac{\frac{n}{n^2 + 1}}{\frac{1}{n}} = \lim_{n \to \infty} \frac{n^2}{n^2 + 1} = \lim_{n \to \infty} \frac{1}{1 + \frac{1}{n^2}} = 1 \neq 0 \]

Therefore, the series in the problem converges if \( \sum \frac{1}{n} \) converges and diverges if \( \sum \frac{1}{n} \) diverges. Since the series \( \sum \frac{1}{n} \) diverges, the series of the problem diverges as well.
3.3 Maclaurin Series

Find the first three nonzero terms in the Maclaurin series for the following functions

1. $y = \cos x$
2. $y = \ln(1 + x)$

Solution

1. Use the formula for the MacLaurin series. The first few derivatives are as follows.

$$f(0) = 1, \quad f'(0) = 0, \quad f''(0) = -1, \quad f'''(0) = 0, \quad f''''(0) = 1$$

So we have the following.

$$\cos x \approx 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$$

2. Use the formula for the MacLaurin series. The first few derivatives are as follows.

$$f(0) = 0, \quad f'(0) = 1, \quad f''(0) = -1, \quad f'''(0) = 2$$

So we have the following.

$$\ln(1 + x) \approx x - \frac{1}{2}x^2 + \frac{1}{3}x^3$$
3.4 Operations with Series

Find the Maclaurin series of the following function.

\[ y = \ln(1 + x^2) \]

**Solution 1**

The following series should be familiar to us already.

\[ \ln(1 + z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n}{n} \]

If we let \( z = x^2 \), we arrive at the following result.

\[ \ln(1 + x^2) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n}}{n} \]

**Solution 2**

Differentiating the function gives us the following.

\[ y' = \frac{2x}{1 + x^2} = \frac{2x}{1 - (-x^2)} \]

Using the formula

\[ \frac{a}{1 - r} = \sum_{n=1}^{\infty} ar^{n-1} \]

we have

\[ y' = \frac{2x}{1 - (-x^2)} = \sum_{n=1}^{\infty} 2x (-x^2)^{n-1} = \sum_{n=1}^{\infty} 2(-1)^{n-1} x^{2n-1} \]

To get back to our function \( y \), we now integrate.

\[ y = \int y' \, dx = \int \sum_{n=1}^{\infty} 2(-1)^{n-1} x^{2n-1} \, dx = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n}}{n} + c \]

Plugging in \( x = 0 \) shows us that

\[ y(0) = \ln(1 + 0^2) = 0, \quad \sum_{n=1}^{\infty} \frac{(-1)^{n-1} 0^{2n}}{n} + c = 0 + c = c \]

So we must have \( c = 0 \). Therefore, we have

\[ \ln(1 + x^2) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n}}{n} \]
3.5 Computations with Series

Use the first 4 terms of the Maclaurin series for \( y = e^{-x} \) to approximate the value of \( e^{-0.2} \). Determine the error of your approximation.

Solution

The first four terms of the Maclaurin series for \( y = e^{-x} \) is the following.

\[
y = e^{-x} \approx 1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3
\]

Plugging in \( x = 0.2 \) gives us

\[
y(-0.2) = e^{-0.2} \approx 1 - 0.2 + \frac{1}{2}(0.2)^2 - \frac{1}{6}(0.2)^3 = 0.8167
\]

The exact value of \( e^{-0.2} \) is approximately 0.81873. Therefore, the error given by our estimate is approximately 0.00006.
3.6 Fourier Series

Determine the Fourier series for the following function on the given interval

\[ f(t) = \begin{cases} 
0 & \text{if } -1 < t \leq 0 \\
 t & \text{if } 0 < t < 1
\end{cases} \]

Solution

First, determine the \( a_0 \) term.

\[ a_0 = \frac{1}{1} \int_{-1}^{1} f(t) \, dt = \int_{0}^{1} t \, dt = \frac{1}{2} t^2 \bigg|_{0}^{1} = \frac{1}{2} \]

Now determine the \( a_n \) terms.

\[ a_n = \frac{1}{1} \int_{-1}^{1} f(t) \cos(nt) \, dt = \int_{0}^{1} t \cos(nt) \, dt = \frac{1}{n} t \sin(nt) + \frac{1}{n^2} \cos(nt) \bigg|_{0}^{1} = \frac{\sin(n)}{n} + \frac{\cos(n)}{n^2} - \frac{1}{n^2} \]

Similarly, determine the \( b_n \) terms.

\[ b_n = \frac{1}{1} \int_{-1}^{1} f(t) \sin(nt) \, dt = \int_{0}^{1} t \sin(nt) \, dt = -\frac{1}{n} t \cos(nt) + \frac{1}{n^2} \sin(nt) \bigg|_{0}^{1} = -\frac{\cos(n)}{n} + \frac{\sin(n)}{n^2} + \frac{1}{n} \]

Then the Fourier series is the following.

\[ f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nt) + b_n \sin(nt) \]

Where \( a_n \) and \( b_n \) are the values found above.
4 First-Order Differential Equations

4.1 Solutions to Differential Equations

Show that the function

\[ y = xe^{-2x} + 3e^{-2x} \]

is a solution to the given differential equation.

\[ \frac{dy}{dx} + 2y = e^{-2x} \]

Solution

To show that the function given is a solution, we first compute its derivative.

\[ y = xe^{-2x} + 3e^{-2x}, \quad y' = -2xe^{-2x} - 5e^{-2x} \]

Now we plug these values (y and y') in our differential equation for y and its derivative.

\[ \left(-2xe^{-2x} - 5e^{-2x}\right) + 2 \left(xe^{-2x} + 3e^{-2x}\right) = -2xe^{-2x} - 5e^{-2x} + 2xe^{-2x} + 6e^{-2x} = e^{-2x} \]

Therefore, our function satisfies the differential equation. This means that the function is a solution.
4.2 Separation of Variables

Find the general solution to the given differential equations

1. \( dx + (2 \cos^2 x - y \cos^2 x) \, dy = 0 \)
2. \( xe^{y} \, dx + e^{-x} \, dy = 0 \)

Solution

1. First, simplify the equation.

\[
\begin{align*}
\quad dx + (2 \cos^2 x - y \cos^2 x) \, dy &= 0 \\
\quad dx + \cos^2 x (2 - y) \, dy &= 0 \\
\quad \sec^2 x \, dx + (2 - y) \, dy &= 0 \\
\end{align*}
\]

Now that the variables are separated (there are no terms with both \( x \) and \( y \)) we can integrate.

\[
\int \sec^2 x \, dx + \int (2 - y) \, dy = c
\]

\[
\tan x + 2y - \frac{1}{2} y^2 = c
\]

This is the general solution to the differential equation.

2. Simplify the equation as before.

\[
\begin{align*}
\quad xe^{y} \, dx + e^{-x} \, dy &= 0 \\
\quad xe^{x} \, dx + e^{-y} \, dy &= 0 \\
\end{align*}
\]

( Divide by \( e^{-x} \) and then by \( e^{y} \) ) Now integrate.

\[
\int xe^{x} \, dx + \int e^{-y} \, dy = c
\]

\[
(x - 1)e^{x} - e^{-y} = c
\]

(We skipped the integration by parts in the first integral) This is the general solution to the differential equation.
4.3 First-Order Linear Differential Equations

Find the solution to the following differential equation.

\[
2 \frac{dy}{dx} - 8xy = e^{2x^2}
\]

**Solution**

First, we need the equation to have no term in front of \( \frac{dy}{dx} \), so we simplify.

\[
2 \frac{dy}{dx} - 8xy = e^{2x^2}
\]

\[
\frac{dy}{dx} - 4xy = \frac{1}{2} e^{2x^2}
\]

Now calculate the integrating factor.

\[\mu(x) = e^{\int -4x \, dx} = e^{-2x^2}\]

Now multiply this to both sides of the equation.

\[e^{-2x^2} \frac{dy}{dx} - 4xe^{-2x^2} y = \frac{1}{2}\]

The reason for this is that now the left side of the equation can be simplified to the derivative of \( y \) times our integrating factor.

\[e^{-2x^2} \frac{dy}{dx} - 4xe^{-2x^2} y = \frac{1}{2}\]

\[\frac{d}{dx} \left( ye^{-2x^2} \right) = \frac{1}{2}\]

Now integrate both sides and simplify.

\[
\int \frac{d}{dx} \left( ye^{-2x^2} \right) \, dx = \int \frac{1}{2} \, dx
\]

\[ye^{-2x^2} = \frac{1}{2} x + c\]

\[y = \frac{1}{2} xe^{2x^2} + ce^{2x^2}\]

This is the general solution to the differential equation.
4.4 Applications of Differential Equations

A bacteria culture is known to increase at a rate proportional to the number of bacteria present. It is observed that the size of the culture triples in 3 hours. After how many hours should it be 10 times as large?

Solution

The information of the problem tells us that

$$\frac{dP}{dt} = kP$$

Where $P$ is the number of bacteria. Let the initial amount of bacteria be $P_0$ (at $t = 0$). Separating variables and integrating shows that we have

$$P(t) = Ce^{kt}$$

Now plug in $t = 0$ to get

$$P(0) = P_0, \quad P(0) = Ce^0 = C$$

So we have

$$P(t) = P_0e^{kt}$$

The problem states that when $t = 3$, $P = 3P_0$ (triples in 3 hours). So we have

$$P(3) = 3P_0 = P_0e^{3k}$$

Solving for $k$ shows that

$$k = \frac{1}{3} \ln 3$$

So we have

$$P(t) = P_0e^{\frac{1}{3}t\ln 3}$$

The problem now is to determine for what value of $t$ gives $P(t) = 10P_0$. Setting $P(t) = 10P_0$ gives us

$$10P_0 = P_0e^{\frac{1}{3}t\ln 3}$$

$$10 = e^{\frac{1}{3}t\ln 3}$$

$$\ln 10 = \frac{1}{3}t\ln 3$$

$$\frac{3\ln 10}{\ln 3} = t$$
5 Higher Order Differential Equations

5.1 Higher-Order Homogeneous Differential Equations

Find the general solution to the given differential equations

1. \( \frac{6d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0 \)
2. \( 2D^2y - 3Dy + y = 0 \)

Solution

1. Set up and solve the auxiliary equation.

\[
6r^2 - r - 2 = 0
\]
\[(2r + 1)(3r - 2) = 0\]
\[r = -\frac{1}{2}, \ r = \frac{2}{3}\]

Then our general solution is the following.

\[y(t) = c_1e^{-\frac{1}{2}t} + c_2e^{\frac{2}{3}t}\]

2. Set up and solve the auxiliary equation as before.

\[
2r^2 - 3r + 1 = 0
\]
\[(2r - 1)(r - 1) = 0\]
\[r = \frac{1}{2}, \ r = 1\]

Then our general solution is the following.

\[y(t) = c_1e^{\frac{1}{2}t} + c_2e^t\]
### 5.2 Auxiliary Equations

Solve the following differential equations.

1. \((D^2 + 25)y = 0\)
2. \((D^2 - 3D + 5)y = 0\)

**Solution**

1. The auxiliary equation is
   \[ r^2 + 25 = 0 \]
   Which has roots \(r = \pm 5i\). Then since these values are complex, the general solution is
   \[ y(t) = c_1 \sin(5t) + c_2 \cos(5t) \]

2. The auxiliary equation is
   \[ r^2 - 3r + 5 = 0 \]
   \[ r = \frac{3 \pm \sqrt{9 - 4(1)(5)}}{2} = \frac{3 \pm i\sqrt{11}}{2} \]
   Since these values are complex, the imaginary parts turn into sines and cosines as follows.
   \[ y(x) = c_1 e^{\frac{3x}{2}} \sin \left( \frac{\sqrt{11}}{2} x \right) + c_2 e^{\frac{3x}{2}} \cos \left( \frac{\sqrt{11}}{2} x \right) \]
5.3 Non-homogeneous Differential Equations

Find the general solution to the given differential equations.

\[(D^2 - D + 2)y = 4e^{3x}\]

Solution

First solve the auxiliary equation to determine the homogeneous solution.

\[r^2 - r + 2 = 0\]
\[r = \frac{1 \pm \sqrt{1 - 4(1)(2)}}{2} = \frac{1 \pm \sqrt{7}}{2}\]

So the homogeneous equation is the following.

\[y_h(x) = c_1 e^{\frac{\sqrt{7}}{2} x} \sin \left(\frac{\sqrt{7}}{2} x\right) + c_2 e^{\frac{\sqrt{7}}{2} x} \cos \left(\frac{\sqrt{7}}{2} x\right)\]

To determine the particular solution, we guess that it has the form \(y_p(t) = Ae^{3x}\). Then we have

\[y_p(x) = Ae^{3x}\]
\[Dy_p(x) = 3Ae^{3x}\]
\[D^2y_p(x) = 9Ae^{3x}\]

Plugging these into our equation, we get the following.

\[9Ae^{3x} - 3Ae^{3x} + 2Ae^{3x} = 4e^{3x}\]
\[8Ae^{3x} = 4e^{3x}\]
\[A = 2\]

Therefore, our particular equation is \(y_p(t) = 2e^{3x}\). Our general solution has the form \(y_g = y_h + y_p\). So we have

\[y_g(t) = c_1 e^{\frac{\sqrt{7}}{2} x} \sin \left(\frac{\sqrt{7}}{2} x\right) + c_2 e^{\frac{\sqrt{7}}{2} x} \cos \left(\frac{\sqrt{7}}{2} x\right) + 2e^{3x}\]
5.4 Applications of Second-Order Equations

A 2 lb weight stretches a spring 6 in. The weight is pushed 7 in above the equilibrium position and released. Find the motion of the weight as a function of time, assuming no damping.

Solution

Using the equations

\[ F = kx, \quad F = mg \]

and using the information of the problem, we have if \( x = 0 \), the force of the spring is equal to the force of gravity, so we can see

\[ 6k = 2(386.09), \quad k = 128.697 \]

Then using the equation for spring position, we have

\[ 2D^2y + 128.697y = 0 \]

(Where \( y \) is distance from equilibrium) Solving this gives us

\[ y(t) = c_1 \sin(64.35t) + c_2 \cos(64.35t) \]

Our initial conditions are when \( t = 0, \ y = 7, \ \dot{y} = 0 \). Plugging these in, we arrive at

\[ y(t) = 7 \cos(64.35t) \]
5.5 Computing the Laplace Transformation

Verify the identity.

\[ L\{\sin at\} = \frac{a}{s^2 + a^2} \]

**Solution**

Begin with the definition of Laplace transformation.

\[ L\{\sin ax\} = \int_0^{\infty} e^{-sx} \sin ax \, dx = \lim_{t \to \infty} \int_0^t e^{-sx} \sin ax \, dx \]

This integral takes a bit of work to solve. Integration by parts is the usual way of doing it, and it will be omitted.

\[
\lim_{t \to \infty} \int_0^t e^{-sx} \sin ax \, dx = \lim_{t \to \infty} \left. \frac{-ae^{-sx} \cos ax - se^{-sx} \sin ax}{s^2 + a^2} \right|_0^t
\]

\[
= \lim_{t \to \infty} \left( \frac{-ae^{-st} \cos at - se^{-st} \sin at}{s^2 + a^2} \right) + \frac{a}{s^2 + a^2}
\]

\[
= \frac{a}{s^2 + a^2}
\]

(You may need to use L'Hospital's rule to evaluate the limit!)
5.6 Computing the Inverse Laplace Transformation

Compute the inverse Laplace transformation of the function.

\[ F(s) = \frac{5s}{s^2 + 6} \]

Solution

We know the following.

\[ L^{-1} \left( \frac{a}{s^2 + a^2} \right) = \sin at, \quad L^{-1} \left( \frac{s}{s^2 + a^2} \right) = \cos at \]

So we would like to rewrite what we have in terms of these formulas. Since there is an \( s \) term in the numerator, we will use the second equation. Note that the 5 constant does not affect the inverse Laplace, so we have

\[ L^{-1} \left( \frac{5s}{s^2 + 6} \right) = 5L^{-1} \left( \frac{s}{s^2 + (\sqrt{6})^2} \right) = 5 \cos \left( \sqrt{6}t \right) \]
Compute the inverse Laplace transformation of the function.

\[ F(s) = \frac{s}{(s - 1)(s + 3)} \]

**Solution**

We will first do partial fractions to make the problem simpler.

\[ \frac{s}{(s - 1)(s + 3)} = \frac{1/4}{s - 1} + \frac{3/4}{s + 3} \]

Then use the rule

\[ L^{-1} \left( \frac{1}{s - a} \right) = e^{at} \]

To obtain

\[ L^{-1} \left( \frac{s}{(s - 1)(s + 3)} \right) = \frac{1}{4} L^{-1} \left( \frac{1}{s - 1} \right) + \frac{3}{4} L^{-1} \left( \frac{1}{s + 3} \right) = \frac{1}{4} e^{t} + \frac{3}{4} e^{-3t} \]
5.7 Solving Differential Equations Using Laplace Transformations

Use Laplace transformations to solve the following differential equation

\[ y'' - 4y' + 4y = e^{3t}, \quad y(0) = 0, \quad y'(0) = -2 \]

**Solution**

First, apply the Laplace transformation to both sides of the equation and simplify.

\[
L(y'') - 4L(y') + 4L(y) = L(e^{3t})
\]

\[
\left(s^2L(y) - sy(0) - y'(0)\right) - 4 \left(sL(y) - y(0)\right) + 4L(y) = \frac{1}{s - 3}
\]

\[
L(y) \left(s^2 - 4s + 4\right) + 2 = \frac{1}{s - 3}
\]

\[
L(y)(s - 2)^2 = \frac{1}{s - 3} - 2
\]

\[
L(y) = \frac{1}{(s - 3)(s - 2)^2} - \frac{2}{(s - 2)^2}
\]

Now we must apply partial fractions to the right side to simplify further.

\[
\frac{1}{(s - 3)(s - 2)^2} = \frac{1}{s - 3} + \frac{-1}{s - 2} + \frac{-1}{(s - 2)^2}
\]

Then use the rules

\[ L^{-1} \left( \frac{1}{s - a} \right) = e^{at}, \quad L^{-1} \left( \frac{n!}{(s - a)^n} \right) = t^n e^{at} \]

To obtain the final answer.

\[
y = L^{-1} \left( \frac{1}{s - 3} \right) - L^{-1} \left( \frac{1}{s - 2} \right) - 3L^{-1} \left( \frac{1}{(s - 2)^2} \right) = e^{3t} - e^{2t} - 3te^{2t}
\]