Contents

1 Limits

1. $\lim_{x \to -1} \frac{x^2 - 1}{x + 1}$
2. $\lim_{x \to \infty} \frac{x^3 + x^2 + 1}{2x^3 + 1}$

Solution

1. First, factor the numerator and simplify.

$$\lim_{x \to -1} \frac{x^2 - 1}{x + 1} = \lim_{x \to -1} \frac{(x-1)(x+1)}{x + 1} = \lim_{x \to -1} (x - 1)$$

Now we can plug in $x = -1$.

$$\lim_{x \to -1} (x - 1) = -2$$

2.

2 Derivatives

Differentiate the following functions using the Four-Step Process.

1. $f(x) = 1 - 4x$
2. $f(x) = x + x^2$

Solution

Recall the Four-Step Process:

1. Replace x by $x + \Delta x$ and y by $y + \Delta y$ in the function $y = f(x)$.
2. Subtract $y = f(x)$ from both sides.
3. Divide both sides of the resulting expression by Δx.
4. Obtain $f'(x)$ by evaluating $\lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta x}$.

1.

$$\begin{aligned} y = 1 - 4x \longrightarrow y + \Delta y &= 1 - 4(x + \Delta x) \\ \Delta y &= 1 - 4(x + \Delta x) - y \\ &= 1 - 4(x + \Delta x) - (1 - 4x) \\ \frac{\Delta y}{\Delta x} &= \frac{\cancel{1 - 4x} - 4\Delta x - \cancel{1 + 4x}}{\Delta x} \\ &= -4 \\ \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} &= f'(x) = -4 \end{aligned}$$

2.

$$\begin{aligned} y = x + x^2 \longrightarrow y + \Delta y &= x + \Delta x + (x + \Delta x)^2 \\ \Delta y &= x + \Delta x + (x + \Delta x)^2 - y \\ &= x + \Delta x + (x + \Delta x)^2 - (x + x^2) \\ &= \cancel{x} + \Delta x + \cancel{x^2} + 2x\Delta x + (\Delta x)^2 - \cancel{x - x^2} \\ \frac{\Delta y}{\Delta x} &= \frac{\Delta x + 2x\Delta x + (\Delta x)^2}{\Delta x} \\ \frac{\Delta y}{\Delta x} &= 1 + 2x + \Delta x \\ \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} &= f'(x) = 1 + 2x \end{aligned}$$

Differentiate

1. $y = \sqrt{5}x^4 - \sqrt{2}x^3 + \pi$
2. $f(q) = \dfrac{e^q + \ln q}{q^2 - 4}$
3. $f(x) = x^2 \arctan(x^2)$
4. $f(x) = \sqrt{x}\sin(x)$
5. $y = \arctan\left(\dfrac{1}{x}\right)$
6. $y = \sqrt{\arccos x}$
7. $y = \ln(\tan x)$

Solution

1. $y' = \sqrt{5}(4x^3) - \sqrt{2}(3x^2) = 4\sqrt{5}x^3 - 3\sqrt{2}x^2$
2. $f'(q) = \dfrac{(q^2-4)(e^q + 1/q) - (e^q + \ln q)(2q)}{(q^2-4)^2}$
3. $f'(x) = (2x)\arctan(x^2) + x^2 \dfrac{1}{1+(x^2)^2}(2x) = 2x\arctan(x^2) + \dfrac{2x^3}{1+x^4}$
4. $f'(x) = \dfrac{1}{2\sqrt{x}}\sin(x) + \sqrt{x}\cos(x)$
5. $y' = \dfrac{1}{1+(1/x)^2}(-1/x^2) = -\dfrac{1}{x^2+1}$
6. $y' = \dfrac{1}{2\sqrt{\arccos x}}\left(-\dfrac{1}{\sqrt{1-x^2}}\right) = -\dfrac{1}{2\sqrt{(1-x^2)\arccos x}}$
7. $y' = \dfrac{\sec^2 x}{\tan x} = \dfrac{\sec^2 x}{\tan x} \cdot \dfrac{\cos^2 x}{\cos^2 x} = \dfrac{1}{\sin x \cos x}$

3 Applications of Derivatives

3.1 Related Rates

Water is poured into a cylindrical tank of radius three feet. Find the instantaneous rate of change of the volume with respect to the depth.

Solution

$$\begin{aligned} V &= \pi r^2 h, \quad r = 3 \\ &= 9\pi h \\ \frac{\mathrm{d}V}{\mathrm{d}h} &= 9\pi \end{aligned}$$

Wheat is poured on the ground at the rate of 12 ft.^{3}/s. The pile forms a cone whose altitude is always three-fourths of the radius. Find the rate at which the altitude is increasing when the altitude is 6.0 ft. The volume of a cone is $V = \frac{\pi r^2 h}{3}$.

Solution

We are given that $\frac{\mathrm{d}V}{\mathrm{d}t} = 12\frac{\text{ft.}^3}{\text{s}}$ and $h = \frac{3}{4}r$, and we are asked to find $\frac{\mathrm{d}h}{\mathrm{d}t}$ when $h = 6$ ft.

$$\begin{aligned} V &= \pi\frac{r^2 h}{3} \\ &= \pi\frac{(4h/3)^2 h}{3} \\ &= \pi\frac{16h^3}{27} \end{aligned}$$

Differentiate implicity with respect to t and plug in $h = 6$,

$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}t} = 12 &= \pi\frac{48h^2}{27}\frac{\mathrm{d}h}{\mathrm{d}t}\bigg|_{h=6} \\ \frac{108}{36\pi} &= \frac{\mathrm{d}h}{\mathrm{d}t} \\ \frac{3}{\pi} &= \frac{\mathrm{d}h}{\mathrm{d}t} \end{aligned}$$

3.2 Tangent and Normal Lines

Find the equation of the tangent and normal lines to the curve $y^2 - \ln^2 x = x$ at $(1, 1)$.

Solution

Differentiate implicitly with respect to x and then plug in $x = y = 1$,

$$\left. 2y\,\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{2\ln x}{x} = 1 \right|_{x=y=1}$$
$$2\,\frac{\mathrm{d}y}{\mathrm{d}x} = 1$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}$$

So the tangent line is,

$$y - 1 = \frac{1}{2}(x - 1)$$

The slope,m', of the normal line must be such that,

$$\frac{1}{2}m' = -1 \implies m' = -2$$

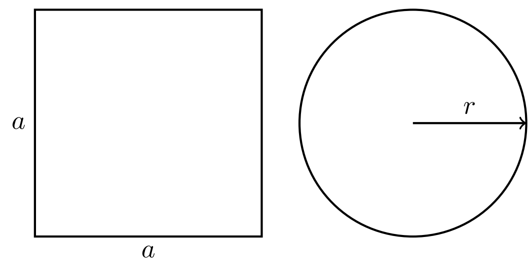
Hence the normal line is,

$$y - 1 = -2(x - 1)$$

3.3 Optimization

A wire of length 50 cm is to be cut into two pieces. On of the pieces is to bent into the form of a circle and the other into the form of a square. How should the wire be cut so that the sum of the enclosed areas is a minimum?

Solution



If a side of the square is length a and the circle is radius r then we are given that

$$50 = 2\pi r + 4a \tag{1}$$

What we want to minimize is the area, A,

$$A = \pi r^2 + a^2 \tag{2}$$

Solve for one of the variables in equation (1) and plug into equation (2),

$$\begin{aligned} r &= \frac{25 - 2a}{\pi} \\ A &= \pi \left(\frac{25 - 2a}{\pi} \right)^2 + a^2 \\ \frac{\mathrm{d}A}{\mathrm{d}a} &= 2\pi \left(\frac{25 - 2a}{\pi} \right) \left(\frac{-2}{\pi} \right) + 2a \end{aligned}$$

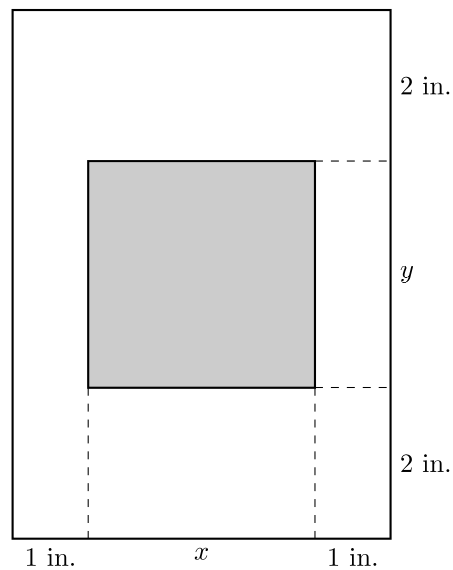
Set this equal to zero and solve for a,

$$\begin{aligned} 0 &= 2\pi \left(\frac{25 - 2a}{\pi} \right) \left(\frac{-2}{\pi} \right) + 2a \\ a &= \frac{50}{\pi + 4} \end{aligned}$$

Plug the value of a into equation (1) to get r,

$$r = \frac{25}{\pi + 4}$$

A rectangular poster is to contain a rectangular picture in the center with an area of 72 in^2. The margins at the top and bottom are to be 2 in. and the left and right margins 1 in. Determine the dimensions of the poster requiring the least amount of material.



We are given that $xy = 72$ and we want to minimize $A = (x+2)(y+4)$. Solve for one of the variables, say y, from the previous equation and plug it into the latter equation.

$$y = \frac{72}{x}$$
$$A = (x+2)\left(\frac{72}{x} + 4\right)$$
$$\frac{\mathrm{d}A}{\mathrm{d}x} = 4 - \frac{144}{x^2}$$

Set this equal to zero and solve for x,

$$x = 6$$
$$y = 12$$

3.4 Differentials

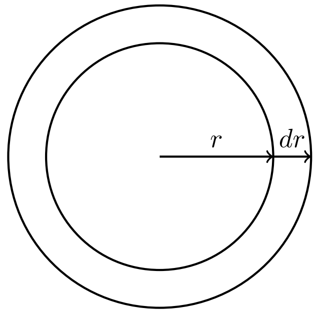
Find dy if $y = x - \sqrt{x}$.

Solution

$$\begin{aligned} dy &= dx + \frac{1}{2\sqrt{x}}dx \\ &= \left(1 + \frac{1}{2\sqrt{x}}\right) dx \end{aligned}$$

Find the approximate formula for the area of a circular ring of radius r and width dr.

Solution



The area of the circle of radius r is πr^2, if the radius is increased by dr then the change in the area is given by the differential dA,

$$A = \pi r^2$$
$$dA = 2\pi r dr$$

This change in area is approximately the area of the ring of width dr.

3.5 Curve Sketching

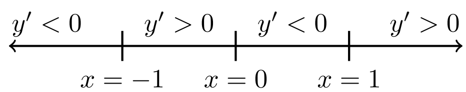
If $y = \dfrac{x^2}{(x^2+1)^2}$ find,

1. the intervals on which y is increasing and decreasing.
2. the extreme values.

Solution

1.

$$\begin{aligned}
y' = 0 &= \frac{(x^2+1)(2x) - x^2(2(x^2+1)(2x)}{(x^2+1)^4} \\
0 &= \frac{2x(x^2+1)^2 - 4x^3(x^2+1)}{(x^2+1)^4} \\
2x(x^2+1) &= 4x^3 \\
0 &= x^3 - x = x(x^2-1) = x(x-1)(x+1) \\
x &= -1, 0, 1
\end{aligned}$$



Thus y is increasing on $(-1,0) \cup (1,\infty)$ and decreasing on $(-\infty,-1) \cup (0,1)$.

2. $y(-1) = \frac{1}{4}$, $y(0) = 0$, $y(1) = \frac{1}{4}$. Hence y has an absolute minimum at $x = 0$ and has absolute maxima at $x = \pm 1$.

If $y = x \arctan(x)$ find,

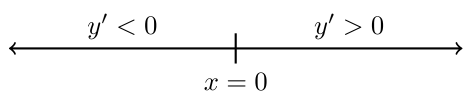
1. the intervals on which y is increasing and decreasing. (*Hint.* There is only one critical point.)
2. the extreme values.
3. the inflection points.
4. the intervals on which y is concave up and concave down.

Sketch the graph of the function.

Solution

1.

$$\begin{aligned} y' = 0 &= \arctan(x) + \frac{x}{1+x^2} \\ x &= 0 \end{aligned}$$



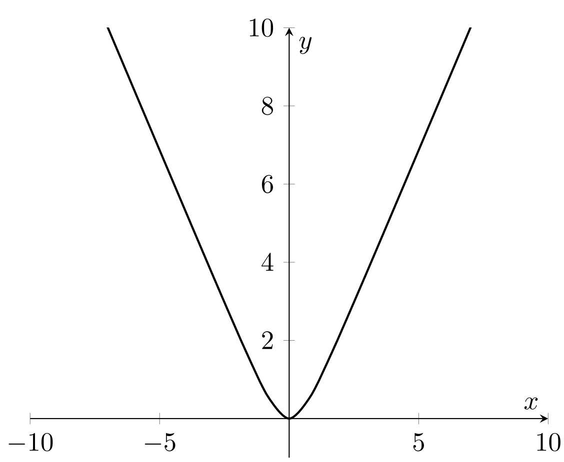
2. y has an absolute minimum at $x = 0$.

3.

$$\begin{aligned} y'' = 0 &= \frac{1}{1+x^2} + \frac{1+x^2-(x)(2x)}{(1+x^2)^2} \\ &= \frac{1}{1+x^2} + \frac{1-x^2}{(1+x^2)^2} \\ &= \frac{2}{(1+x^2)^2} \end{aligned}$$

There are no inflection points because this equation has no solutions.

4. Since $y'' > 0$ for all x then the graph is always concave upwards.



4 Integrals

Integrate the following functions.

1. $\displaystyle\int \left(x^3 - \frac{2}{x^7}\right)\,dx$
2. $\displaystyle\int x^3\sqrt{x^2-3}\,dx$
3. $\displaystyle\int \frac{dx}{x\ln x}$
4. $\displaystyle\int \frac{x^3\,dx}{\sqrt{x^4+4}}$

Solution

1.

$$\begin{aligned}\int \left(x^3 - \frac{2}{x^7}\right)\,dx &= \frac{x^4}{4} - 2\frac{x^{-6}}{-6} + C\\ &= \frac{x^4}{4} - \frac{1}{3x^6} + C\end{aligned}$$

2.

$$\begin{aligned}&\int x^3\sqrt{x^2-3}\,dx \qquad \begin{cases} u = x^2 - 3 & \longrightarrow x^2 = u + 3\\ du = 2x\,dx \end{cases}\\ &= \frac{1}{2}\int \sqrt{u}(x+3)\,du\\ &= \frac{1}{2}\int (u^{3/2} + 3u^{1/2})\,du\\ &= \frac{1}{2}\left(\frac{u^{5/2}}{5/2} + 3\frac{u^{3/2}}{3/2} + C\right)\Bigg|_{u=x^2-3}\\ &= 5(x^2-3)^{5/2} + 2(x^2-3)^{3/2} + C\end{aligned}$$

3.

$$\begin{aligned}&\int \frac{dx}{x\ln x} \qquad \begin{cases} u = \ln x\\ du = dx/x \end{cases}\\ &= \int \frac{du}{u}\\ &= (\ln u + C)\big|_{u=\ln x}\\ &= \ln(\ln x) + C\end{aligned}$$

4.

$$\int \frac{x^3\,dx}{\sqrt{x^4+4}} \qquad \begin{cases} u = x^4+4 \\ du = 4x^3\,dx \end{cases}$$

$$= \frac{1}{4}\int \frac{du}{\sqrt{u}}$$

$$= \frac{1}{4}\left(\frac{u^{1/2}}{1/2}\right)\bigg|_{u=x^4+4} + C$$

$$= \frac{1}{2}\sqrt{x^4+4} + C$$

4.1 Improper Integrals

Evaluate the following improper integrals.

1. $\displaystyle\int_{-\infty}^{0} \frac{2x\,dx}{(x^2+4)^2}$
2. $\displaystyle\int_{1}^{\infty} \frac{dx}{x^2}$
3. $\displaystyle\int_{4}^{5} \frac{dx}{x-4}$

Solution

1.

$$\begin{aligned}
&\int_{-\infty}^{0} \frac{2x\,dx}{(x^2+4)^2} \qquad \begin{cases} u = x^2+4 & u : \infty \to 4 \\ du = 2x\,dx \end{cases} \\
&= \int_{\infty}^{4} \frac{du}{u^2} \\
&= -\frac{1}{u}\bigg|_{\infty}^{4} \\
&= -\frac{1}{4}
\end{aligned}$$

2.

$$\begin{aligned}
\int_{1}^{\infty} \frac{dx}{x^2} &= -\frac{1}{x^2}\bigg|_{1}^{\infty} \\
&= 1
\end{aligned}$$

3.

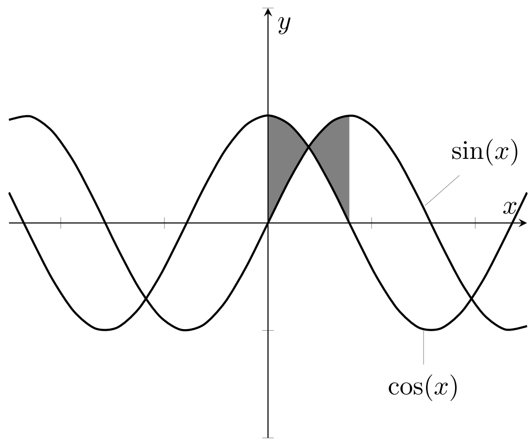
$$\begin{aligned}
\int_{4}^{5} \frac{dx}{x-4} &= \ln|x-4|\big|_{4}^{5} \\
&= \ln 1 - (-\infty) \\
&\to \infty
\end{aligned}$$

5 Applications of Integrals

5.1 Areas between Functions

Find the area bounded by $y = \sin x$ and $y = \cos x$ when $0 \leq x \leq \pi/2$.

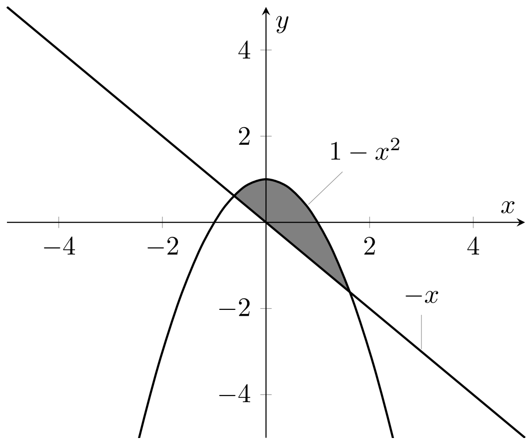
Solution



$$
\begin{aligned}
A &= \int_0^{\pi/4} (\cos x - \sin x)\, dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\, dx \\
&= (\sin x + \cos x)\Big|_0^{\pi/4} + (-\cos x - \sin x)\Big|_{\pi/4}^{\pi/2} \\
&= \left(2\frac{\sqrt{2}}{2} - 1\right) - \left(1 - 2\frac{\sqrt{2}}{2}\right) \\
&= 2\sqrt{2} - 2
\end{aligned}
$$

Set up but do not evaluate the integral that expresses the area bounded by $y = 1 - x^2$ and $y = -x$.

Solution



First find where the two curves intersect,

$$\begin{aligned} -x &= 1 - x^2 \\ x^2 - x - 1 &= 0 \\ x &= \frac{1 \pm \sqrt{1 - 4(1)(-1)}}{2} \\ &= \frac{1 \pm \sqrt{5}}{2} \end{aligned}$$

Then the area, A, is,

$$A = \int_{\frac{1}{2}(1-\sqrt{5})}^{\frac{1}{2}(1+\sqrt{5})} (1 - x^2 - x)\, dx$$

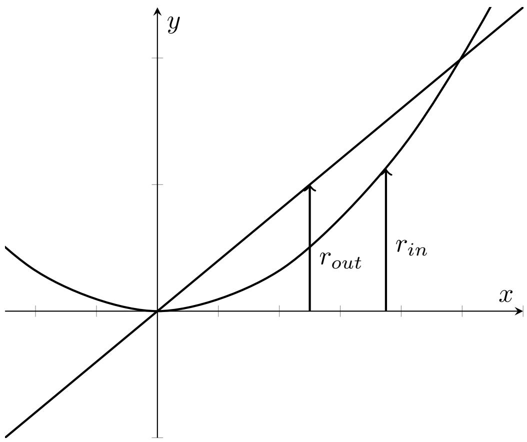
5.2 Volumes of Revolution

A region A is bounded by $y = x$ and $y = x^2$ from $x = 0$ to $x = 1$.

1. Calculate the volume generated from rotating A about the x-axis.
2. Calculate the volume generate from rotating A about the y-axis.

Solution

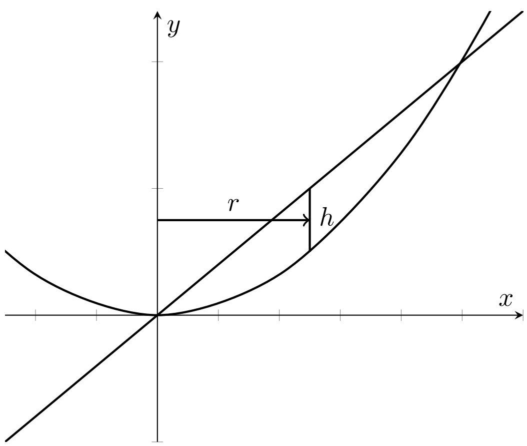
1. The method of washers will be used.



From the graph it is seen that $r_{out} = x$ and $r_{in} = x^2$. Then the volume is,

$$
\begin{aligned}
V &= \pi \left(\int_0^1 x^2 \, dx - \int_0^1 x^4 \, dx \right) \\
&= \pi \left(\frac{1}{3} - \frac{1}{5} \right) \\
&= \frac{2\pi}{15}
\end{aligned}
$$

2. The method of shells will be used.

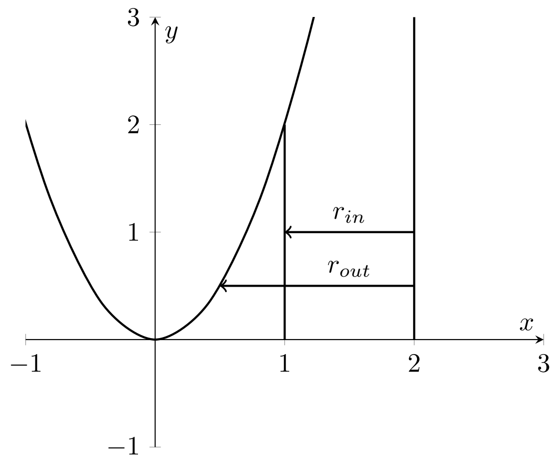


From the graph it is seen that $r = x$ and $h = x - x^2$. Hence the volume is,

$$
\begin{aligned}
V &= 2\pi \int_0^1 x(x - x^2) \, dx \\
&= 2\pi \left(\frac{x^3}{3} - \frac{x^4}{4} \right) \Big|_0^1 \\
&= \frac{\pi}{6}
\end{aligned}
$$

Find the volume of the solid generated by revolving the region bounded by $y = 2x^2$, $x = 1$, and the x-axis about the line $x = 2$.

Solution



Using the method of washers, we see that $r_{in} = 1$ and $r_{out} = 2 - x = 2 - \sqrt{\frac{y}{2}}$. Hence the volume is,

$$
\begin{aligned}
V &= \pi\left(\int_0^2 \left(2 - \sqrt{\frac{y}{2}}\right)^2 dy - \int_0^2 1\, dy\right) \\
&= \pi\left(\int_0^2 4 - 4\sqrt{\frac{y}{2}} + \frac{y}{2}\, dy - 2\right) \\
&= \pi\left(4y - 2\sqrt{2}\frac{y^{3/2}}{3/2} + \frac{y^2}{4} - 2\right)\Bigg|_0^2 \\
&= \pi\left(8 - \frac{4\sqrt{2}}{3}2\sqrt{2} + 1 - 2\right) \\
&= \pi\left(7 - \frac{2(2\sqrt{2})^2}{3}\right) \\
&= \frac{5\pi}{3}
\end{aligned}
$$