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1 Limits

1. \( \lim_{x \to -1} \frac{x^2 - 1}{x + 1} \)

2. \( \lim_{x \to \infty} \frac{x^3 + x^2 + 1}{2x^3 + 1} \)

Solution

1. First, factor the numerator and simplify.

\[
\lim_{x \to -1} \frac{x^2 - 1}{x + 1} = \lim_{x \to -1} \frac{(x - 1)(x + 1)}{x + 1} = \lim_{x \to -1} (x - 1)
\]

Now we can plug in \( x = -1 \).

\[
\lim_{x \to -1} (x - 1) = -2
\]

2.
2 Derivatives

Differentiate the following functions using the Four-Step Process.

1. \( f(x) = 1 - 4x \)
2. \( f(x) = x + x^2 \)

Solution

Recall the Four-Step Process:

1. Replace \( x \) by \( x + \Delta x \) and \( y \) by \( y + \Delta y \) in the function \( y = f(x) \).
2. Subtract \( y = f(x) \) from both sides.
3. Divide both sides of the resulting expression by \( \Delta x \).
4. Obtain \( f'(x) \) by evaluating \( \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \).

1. 
   
   \[
   y = 1 - 4x \quad \rightarrow \quad y + \Delta y = 1 - 4(x + \Delta x)
   \]
   
   \[
   \Delta y = 1 - 4(x + \Delta x) - y
   \]
   
   \[
   = 1 - 4(x + \Delta x) - (1 - 4x)
   \]
   
   \[
   \frac{\Delta y}{\Delta x} = \frac{1 - 4x - 4\Delta x - 1 + 4x}{\Delta x}
   \]
   
   \[
   = -4
   \]
   
   \[
   \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x) = -4
   \]

2. 
   
   \[
   y = x + x^2 \quad \rightarrow \quad y + \Delta y = x + \Delta x + (x + \Delta x)^2
   \]
   
   \[
   \Delta y = x + \Delta x + (x + \Delta x)^2 - y
   \]
   
   \[
   = x + \Delta x + (x + \Delta x)^2 - (x + x^2)
   \]
   
   \[
   = \Delta x + \Delta x^2 + 2x\Delta x + (\Delta x)^2 - x - x^2
   \]
   
   \[
   \frac{\Delta y}{\Delta x} = \frac{\Delta x + 2x\Delta x + (\Delta x)^2}{\Delta x}
   \]
   
   \[
   \frac{\Delta y}{\Delta x} = 1 + 2x + \Delta x
   \]
   
   \[
   \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x) = 1 + 2x
   \]
Differentiate

1. \( y = \sqrt{5}x^4 - \sqrt{2}x^3 + \pi \)
2. \( f(q) = \frac{e^q + \ln q}{q^2 - 4} \)
3. \( f(x) = x^2 \arctan(x^2) \)
4. \( f(x) = \sqrt{x} \sin(x) \)
5. \( y = \arctan\left(\frac{1}{x}\right) \)
6. \( y = \sqrt{\arccos x} \)
7. \( y = \ln(\tan x) \)

Solution

1. \( y' = \sqrt{5}(4x^3) - \sqrt{2}(3x^2) = 4\sqrt{5}x^3 - 3\sqrt{2}x^2 \)
2. \( f'(q) = \frac{(q^2 - 4)(e^q + 1/q) - (e^q + \ln q)(2q)}{(q^2 - 4)^2} \)
3. \( f'(x) = (2x) \arctan(x^2) + x^2 \frac{1}{1 + (x^2)^2} (2x) = 2x \arctan(x^2) + \frac{2x^3}{1 + x^4} \)
4. \( f'(x) = \frac{1}{2\sqrt{x}} \sin(x) + \sqrt{x} \cos(x) \)
5. \( y' = \frac{1}{1 + (1/x)^2}(-1/x^2) = -\frac{1}{x^2 + 1} \)
6. \( y' = \frac{1}{2\sqrt{\arccos x}} \left( -\frac{1}{\sqrt{1 - x^2}} \right) = -\frac{1}{2\sqrt{(1 - x^2)^{\arccos x}}} \)
7. \( y' = \frac{\sec^2 x}{\tan x} = \frac{\sec^2 x}{\tan x} \cdot \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\sin x \cos x} \)
3 Applications of Derivatives

3.1 Related Rates

Water is poured into a cylindrical tank of radius three feet. Find the instantaneous rate of change of the volume with respect to the depth.

Solution

\[ V = \pi r^2 h, \quad r = 3 \]
\[ = 9\pi h \]
\[ \frac{dV}{dh} = 9\pi \]
Wheat is poured on the ground at the rate of 12 ft.\(^3\)/s. The pile forms a cone whose altitude is always three-fourths of the radius. Find the rate at which the altitude is increasing when the altitude is 6.0 ft. The volume of a cone is \( V = \frac{\pi r^2 h}{3} \).

**Solution**

We are given that \( \frac{dV}{dt} = 12 \text{ ft.}^3/\text{s} \) and \( h = \frac{3}{4} r \), and we are asked to find \( \frac{dh}{dt} \) when \( h = 6 \) ft.

\[
V = \frac{\pi r^2 h}{3} \\
= \frac{\pi (4h/3)^2 h}{3} \\
= \frac{\pi 16h^3}{27}
\]

Differentiate implicitly with respect to \( t \) and plug in \( h = 6 \),

\[
\frac{dV}{dt} = 12 = \frac{\pi 48h^2}{27} \frac{dh}{dt} \bigg|_{h=6} \\
\frac{108}{36\pi} = \frac{dh}{dt} \\
\frac{3}{\pi} = \frac{dh}{dt}
\]
3.2 Tangent and Normal Lines

Find the equation of the tangent and normal lines to the curve \( y^2 - \ln^2 x = x \) at \((1, 1)\).

**Solution**

Differentiate implicitly with respect to \( x \) and then plug in \( x = y = 1 \),

\[
2y \frac{dy}{dx} - \frac{2 \ln x}{x} = 1 \bigg|_{x=y=1}
\]

\[
2 \frac{dy}{dx} = 1
\]

\[
\frac{dy}{dx} = \frac{1}{2}
\]

So the tangent line is,

\[
y - 1 = \frac{1}{2}(x - 1)
\]

The slope, \( m' \), of the normal line must be such that,

\[
\frac{1}{2} m' = -1 \implies m' = -2
\]

Hence the normal line is,

\[
y - 1 = -2(x - 1)
\]
3.3 Optimization

A wire of length 50 cm is to be cut into two pieces. One of the pieces is to be bent into the form of a circle and the other into the form of a square. How should the wire be cut so that the sum of the enclosed areas is a minimum?

Solution

If a side of the square is length $a$ and the circle is radius $r$ then we are given that

$$50 = 2\pi r + 4a \quad (1)$$

What we want to minimize is the area, $A$,

$$A = \pi r^2 + a^2 \quad (2)$$

Solve for one of the variables in equation (1) and plug into equation (2),

$$r = \frac{25 - 2a}{\pi}$$

$$A = \pi \left( \frac{25 - 2a}{\pi} \right)^2 + a^2$$

$$\frac{dA}{da} = 2\pi \left( \frac{25 - 2a}{\pi} \right) \left( -\frac{2}{\pi} \right) + 2a$$

Set this equal to zero and solve for $a$,

$$0 = 2\pi \left( \frac{25 - 2a}{\pi} \right) \left( -\frac{2}{\pi} \right) + 2a$$

$$a = \frac{50}{\pi + 4}$$

Plug the value of $a$ into equation (1) to get $r$,

$$r = \frac{25}{\pi + 4}$$
A rectangular poster is to contain a rectangular picture in the center with an area of 72 in\(^2\). The margins at the top and bottom are to be 2 in. and the left and right margins 1 in. Determine the dimensions of the poster requiring the least amount of material.

We are given that \(xy = 72\) and we want to minimize \(A = (x + 2)(y + 4)\). Solve for one of the variables, say \(y\), from the previous equation and plug it into the latter equation.

\[
y = \frac{72}{x}
\]

\[
A = (x + 2) \left( \frac{72}{x} + 4 \right)
\]

\[
\frac{dA}{dx} = 4 - \frac{144}{x^2}
\]

Set this equal to zero and solve for \(x\),

\[
x = 6
\]

\[
y = 12
\]
3.4 Differentials

Find $dy$ if $y = x - \sqrt{x}$.

Solution

\[
dy = dx + \frac{1}{2\sqrt{x}} dx
\]

\[
= \left( 1 + \frac{1}{2\sqrt{x}} \right) dx
\]
Find the approximate formula for the area of a circular ring of radius $r$ and width $dr$.

**Solution**

The area of the circle of radius $r$ is $\pi r^2$, if the radius is increased by $dr$ then the change in the area is given by the differential $dA$,

\[
A = \pi r^2 \\
\frac{dA}{dr} = 2\pi r \\
dA = 2\pi r dr
\]

This change in area is approximately the area of the ring of width $dr$. 

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$\text{22100 Exam Jam}$
3.5 Curve Sketching

If \( y = \frac{x^2}{(x^2 + 1)^2} \) find,

1. the intervals on which \( y \) is increasing and decreasing.

2. the extreme values.

Solution

1. 

\[
y' = 0 = \frac{(x^2 + 1)(2x) - x^2(2(x^2 + 1))(2x)}{(x^2 + 1)^4}
= \frac{2x(x^2 + 1)^2 - 4x^3(x^2 + 1)}{(x^2 + 1)^4}
= 2x(x^2 + 1) = 4x^3
\]

\[
0 = x^3 - x = x(x^2 - 1) = x(x - 1)(x + 1)
\]

\[
x = -1, 0, 1
\]

\[
y' < 0 \quad y' > 0 \quad y' < 0 \quad y' > 0
\]

\[
x = -1 \quad x = 0 \quad x = 1
\]

Thus \( y \) is increasing on \((-1, 0) \cup (1, \infty)\) and decreasing on \((-\infty, -1) \cup (0, 1)\).

2. \( y(-1) = \frac{1}{4}, \ y(0) = 0, \ y(1) = \frac{1}{4} \). Hence \( y \) has an absolute minimum at \( x = 0 \) and has absolute maxima at \( x = \pm1 \).
If $y = x \arctan(x)$ find,

1. the intervals on which $y$ is increasing and decreasing. (Hint. There is only one critical point.)

2. the extreme values.

3. the inflection points.

4. the intervals on which $y$ is concave up and concave down.

Sketch the graph of the function.

**Solution**

1. 

$$y' = 0 = \arctan(x) + \frac{x}{1 + x^2}$$

$$x = 0$$

$$y' < 0 \quad \quad \quad y' > 0$$

2. $y$ has an absolute minimum at $x = 0$.

3. 

$$y'' = 0 = \frac{1}{1 + x^2} + \frac{1 + x^2 - (x)(2x)}{(1 + x^2)^2}$$

$$= \frac{1}{1 + x^2} + \frac{1 - x^2}{(1 + x^2)^2}$$

$$= \frac{2}{(1 + x^2)^2}$$

There are no inflection points because this equation has no solutions.

4. Since $y'' > 0$ for all $x$ then the graph is always concave upwards.
4 Integrals

Integrate the following functions.

1. \( \int \left( x^3 - \frac{2}{x^7} \right) \, dx \)

2. \( \int x^3 \sqrt{x^2 - 3} \, dx \)

3. \( \int \frac{dx}{x \ln x} \)

4. \( \int \frac{x^3 \, dx}{\sqrt{x^4 + 4}} \)

Solution

1. 
   \[
   \int \left( x^3 - \frac{2}{x^7} \right) \, dx = \frac{x^4}{4} - 2\frac{x^{-6}}{-6} + C \\
   = \frac{x^4}{4} - \frac{1}{3x^6} + C
   
   \]

2. 
   \[
   \int x^3 \sqrt{x^2 - 3} \, dx \quad \left\{ \begin{array}{l}
   u = x^2 - 3 \\
   du = 2x \, dx
   \end{array} \right. \\
   = \frac{1}{2} \int \sqrt{u}(x + 3) \, du \\
   = \frac{1}{2} \int (u^{3/2} + 3u^{1/2}) \, du \\
   = \frac{1}{2} \left( \frac{u^{5/2}}{5/2} + 3\frac{u^{3/2}}{3/2} + C \right) \bigg|_{u=x^2-3} \\
   = 5(x^2 - 3)^{5/2} + 2(x^2 - 3)^{3/2} + C
   
   \]

3. 
   \[
   \int \frac{dx}{x \ln x} \quad \left\{ \begin{array}{l}
   u = \ln x \\
   du = dx/x
   \end{array} \right. \\
   = \int \frac{du}{u} \\
   = (\ln u + C) \bigg|_{u=\ln x} \\
   = \ln(\ln x) + C
   
   \]
4.

\[
\int \frac{x^3 \, dx}{\sqrt{x^4 + 4}} \quad \begin{cases} 
  u = x^4 + 4 \\
  du = 4x^3 \, dx
\end{cases}
\]

\[
= \frac{1}{4} \int \frac{du}{\sqrt{u}}
\]

\[
= \frac{1}{4} \left( u^{1/2} \right) \bigg|_{u = x^4 + 4} + C
\]

\[
= \frac{1}{2} \sqrt{x^4 + 4} + C
\]
4.1 Improper Integrals

Evaluate the following improper integrals.

1. \[ \int_{-\infty}^{0} \frac{2x}{(x^2 + 4)^2} \, dx \]
2. \[ \int_{1}^{\infty} \frac{dx}{x^2} \]
3. \[ \int_{4}^{5} \frac{dx}{x - 4} \]

Solution

1.
\[ \int_{-\infty}^{0} \frac{2x}{(x^2 + 4)^2} \, dx \quad \left\{ \begin{array}{l}
u = x^2 + 4 \quad u : \infty \to 4 \\
du = 2x \, dx \end{array} \right. \\
= \int_{\infty}^{4} \frac{du}{u^2} \\
= - \frac{1}{u} \bigg|_{\infty}^{4} \\
= - \frac{1}{4} \]

2.
\[ \int_{1}^{\infty} \frac{dx}{x^2} = - \frac{1}{x^2} \bigg|_{1}^{\infty} = 1 \]

3.
\[ \int_{4}^{5} \frac{dx}{x - 4} = \ln |x - 4| \bigg|_{4}^{5} = \ln 1 - (-\infty) \to \infty \]
5 Applications of Integrals

5.1 Areas between Functions

Find the area bounded by \( y = \sin x \) and \( y = \cos x \) when \( 0 \leq x \leq \pi/2 \).

Solution

\[
A = \int_0^{\pi/4} (\cos x - \sin x) \, dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x) \, dx \\
= (\sin x + \cos x) \bigg|_0^{\pi/4} + (-\cos x - \sin x) \bigg|_{\pi/4}^{\pi/2} \\
= \left(2 \cdot \frac{\sqrt{2}}{2} - 1\right) - \left(1 - 2 \cdot \frac{\sqrt{2}}{2}\right) \\
= 2\sqrt{2} - 2
\]
Set up but do not evaluate the integral that expresses the area bounded by \( y = 1 - x^2 \) and \( y = -x \).

**Solution**

First find where the two curves intersect,

\[-x = 1 - x^2\]
\[x^2 - x - 1 = 0\]
\[x = \frac{1 \pm \sqrt{1 - 4(1)(-1)}}{2}\]
\[= \frac{1 \pm \sqrt{5}}{2}\]

Then the area, \( A \), is,

\[A = \int_{\frac{1}{2}(1 + \sqrt{5})}^{\frac{1}{2}(1 - \sqrt{5})} (1 - x^2 - x) \, dx\]
5.2 Volumes of Revolution

A region $A$ is bounded by $y = x$ and $y = x^2$ from $x = 0$ to $x = 1$.

1. Calculate the volume generated from rotating $A$ about the $x$-axis.

2. Calculate the volume generated from rotating $A$ about the $y$-axis.

Solution

1. The method of washers will be used.

   From the graph it is seen that $r_{\text{out}} = x$ and $r_{\text{in}} = x^2$. Then the volume is,

   $V = \pi \left( \int_0^1 x^2 \, dx - \int_0^1 x^4 \, dx \right)$

   $= \pi \left( \frac{1}{3} - \frac{1}{5} \right)$

   $= \frac{2\pi}{15}$

2. The method of shells will be used.

   From the graph it is seen that $r = x$ and $h = x - x^2$. Hence the volume is,

   $V = 2\pi \int_0^1 x(x - x^2) \, dx$

   $= 2\pi \left( \frac{x^3}{3} - \frac{x^4}{4} \right) \bigg|_0^1$

   $= \frac{\pi}{6}$
Find the volume of the solid generated by revolving the region bounded by $y = 2x^2$, $x = 1$, and the $x$-axis about the line $x = 2$.

**Solution**

\[
V = \pi \left( \int_0^2 \left( 2 - \sqrt{\frac{y}{2}} \right)^2 dy - \int_0^2 1 dy \right)
\]

\[
= \pi \left( \int_0^2 \left( 4 - 4\sqrt{\frac{y}{2}} + \frac{y}{2} dy - 2 \right) \right)
\]

\[
= \pi \left( 4y - 2\sqrt{2}y^{3/2} + \frac{y^2}{4} - 2 \right) \bigg|_0^2
\]

\[
= \pi \left( 8 - \frac{4\sqrt{2}}{3}2\sqrt{2} + 1 - 2 \right)
\]

\[
= \pi \left( 7 - \frac{2(2\sqrt{2})^2}{3} \right)
\]

\[
= \frac{5\pi}{3}
\]