Contents

1 Geometry of $\mathbb{R}^2$

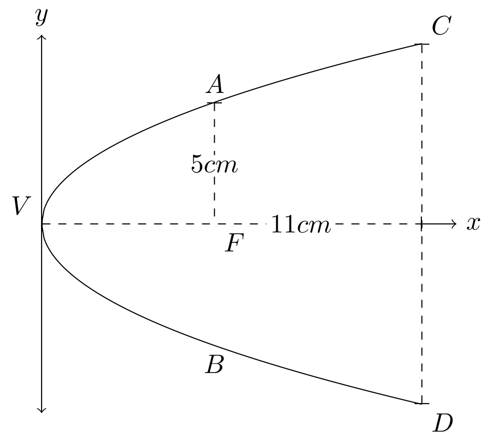
1.1 Conic Sections

A cross-section of a parabolic reflector is shown in the figure. The bulb is located at the focus and the opening at the focus is 10 cm.

(a) Find an equation of the parabola.

(b) Find the diameter of the opening $|CD|$, 11 cm from the vertex.

(c) Parametrize the bottom half of the parabola.



Solution

To begin solving part (a), we start with the general equation for a horizontal parabola centered at the origin (for simplicity) and noticing that $F : (p, 0)$

$$y^2 = 4px, \quad \text{where p is the distance from the focus to the vertex}$$
$$\text{When } x = p,\ y = \pm 5 \text{ (From figure)} \implies (\pm 5)^2 = 4p(p) = 4p^2$$
$$\implies p = \frac{5}{2} \implies \boxed{y^2 = 10x}$$

As for part (b), we set $x = 11$ and see that $y = \pm\sqrt{110}$. Thus, the distance $\boxed{|CD| = 2\sqrt{110}}$

Finally, for part (c) we see $x = t$, our parameter and see that $y^2 = 10t \implies y = -\sqrt{10t}$ taking only the negative value of the square root because the goal is to parametrize the bottom half of the parabola (where $y < 0$). Thus, our parametrization is:

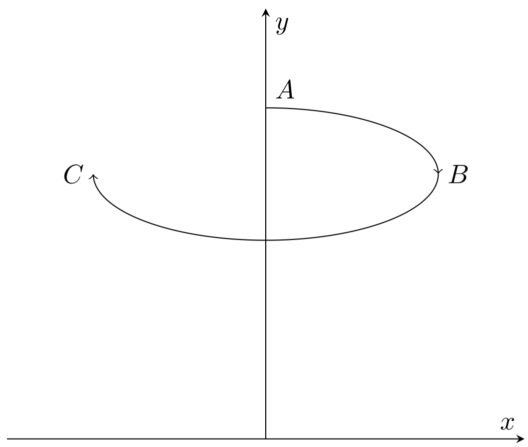
$$\boxed{x = t, \quad y = -\sqrt{10t}}$$

1.2 Parametric Equations

By first eliminating the parameter, describe the motion of a particle following the graph of the parametric equations:

$$x = 2\sin t, \quad y = 4 + \cos t, \quad 0 \le t \le \frac{3\pi}{2}$$

Solution



Here, to see how we arrived at the graph shown above, we solve for x, y in terms of $\sin t$ and $\cos t$ and use the Pythagorean identity as follows:

$$\frac{x}{2} = \sin t, \quad y - 4 = \cos t$$
$$\implies \frac{x^2}{4} + (y-4)^2 = 1$$

The final equation describes the ellipse shown. Furthermore, plugging in the values $t = 0,\ t = \frac{3\pi}{2}$ we find out that the particle travels from point A to point C through the point B ($t = \frac{\pi}{2}$)

1.3 More Parametric Equations

Find the Cartesian equations for each of the curves described by the parametric equations below:

(a) $x = e^t - 1, \quad y = e^{2t}$

(b) $x = \tan^2\theta, \quad y = \sec\theta, \quad \frac{-\pi}{2} < \theta < \frac{\pi}{2}$

Solution

To obtain the Cartesian equation for part (a), we will begin by solving for e^t in terms of x:

$$x = e^t - 1 \implies e^t = x + 1$$
$$y = e^{2t} = (e^t)^2 \implies \boxed{y = (x+1)^2}$$

As for part (b), we will use a variation of the Pythagorean Theorem:

$$1 + \tan^2\theta = \sec^2\theta \implies 1 + x = y^2$$

Here, to solve for y we should recall that the cosine function is positive in the specified interval, which implies that the secant is also positive in $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$. This means that we need only take the positive square root to arrive at the correct equation:

$$\boxed{y = \sqrt{1+x}}$$

1.4 Polar Coordinates

Sketch the curve

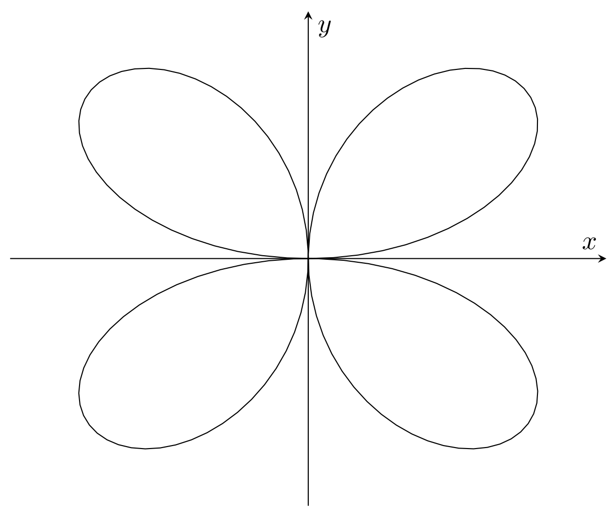
$$(x^2 + y^2)^3 = 4x^2y^2$$

Solution

As the title suggests, switching to polar coordinates will make the problem much simpler:

$$\begin{aligned}
&x = r\cos\theta, \quad y = r\sin\theta \\
&\implies (r^2)^3 = 4r^4\cos^2\theta\sin^2\theta \\
&\implies r^2 = \sin^2 2\theta \text{ (Using the double-angle formula for sines)} \\
&\implies r = |\sin 2\theta|
\end{aligned}$$

Notice that the cases $r = \sin 2\theta$ and $r = -\sin 2\theta$ overlap since the graph of $r = \sin 2\theta$ is symmetric about the origin. Thus, the sketch is as shown (traced twice).



2 Complex Numbers

2.1 Polar Form

Use Euler's formula to prove that:

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

Solution

As is suggested, we will used the formula:

$$\begin{aligned}
&e^{ix} = \cos x + i \sin x \\
&\implies e^{-ix} = \cos(-x) + i\sin(-x) = \cos x - i \sin x \\
&\text{(Since } \cos x \text{ is an even function, } \sin x \text{ an odd one)} \\
&e^{ix} + e^{-ix} = 2\cos x \implies \boxed{\cos x = \frac{e^{ix} + e^{-ix}}{2}}
\end{aligned}$$

In a similar fashion, we can derive the formula for $\sin x$ by subtracting the two equations:

$$e^{ix} - e^{-ix} = 2i \sin x \implies \boxed{\sin x = \frac{e^{ix} - e^{-ix}}{2i}}$$

2.2 Roots of Unity

Find all the cube roots of 1 and show that if one of the complex roots (i.e. nonzero imaginary part) is labeled z then the other complex root is z^2.

Solution

Here, we will begin with De Moivre's Theorem:

$$\omega_k = r^{\frac{1}{n}} \left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i \sin\left(\frac{\theta + 2k\pi}{n}\right) \right] \quad \text{for } k = 0, 1, .., n-1$$

In our case, we see that $n = 3,\ \theta = 0$ (1 is on the positive real axis), $r = |1| = 1 \implies r^{\frac{1}{3}} = 1$ and $k = 0, 1, 2$:

$$\begin{aligned} \omega_0 &= \cos 0 + i \sin 0 = \boxed{1} \\ \omega_1 &= \cos\frac{2\pi}{3} + i \sin\frac{2\pi}{3} = \boxed{-\frac{1}{2} + \frac{\sqrt{3}}{2} i} \\ \omega_2 &= \cos\frac{4\pi}{3} + i \sin\frac{4\pi}{3} = \boxed{-\frac{1}{2} - \frac{\sqrt{3}}{2} i} \end{aligned}$$

Now, suppose we name $z = \omega_1 = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$:

$$\begin{aligned} z^2 &= \left(-\frac{1}{2} + \frac{\sqrt{3}}{2}\right)^2 = \left(\frac{1}{2}\right)^2 - 2 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} i + \left(\frac{\sqrt{3}}{2} i\right)^2 = \left(\frac{1}{4} - \frac{3}{4}\right) - \frac{\sqrt{3}}{2} i \\ \implies z^2 &= -\frac{1}{2} - \frac{\sqrt{3}}{2} i = \omega_2 \implies \boxed{(\omega_1)^2 = \omega_2} \end{aligned}$$

Notice that if we had chosen $z = \omega_2$ a similar calculation works similarly:

$$\begin{aligned} z^2 &= \left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\right)^2 = \left(\frac{1}{2}\right)^2 + 2 \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} i + \left(\frac{\sqrt{3}}{2} i\right)^2 = \left(\frac{1}{4} - \frac{3}{4}\right) + \frac{\sqrt{3}}{2} i \\ \implies z^2 &= -\frac{1}{2} + \frac{\sqrt{3}}{2} i = \omega_1 \implies \boxed{(\omega_2)^2 = \omega_1} \end{aligned}$$

3 Geometry of $\mathbb{R}^3$

3.1 The Distance Formula

Describe the set of points P such that the distance from P to $A(-1,5,3)$ is equal to the distance from P to $B(6,2,-2)$.

Solution

To understand the set at hand, we will first determine an equation which all such points P must satisfy. To do so, we play with the distance formula:

$$\begin{aligned} d(P,A) &= d(P,B) \\ \sqrt{(x+1)^2+(y-5)^2+(z-3)^2} &= \sqrt{(x-6)^2+(y-2)^2+(z+2)^2} \\ (x+1)^2+(y-5)^2+(z-3)^2 &= (x-6)^2+(y-2)^2+(z+2)^2 \\ x^2+2x+1+y^2-10y+25+z^2-6z+9 &= x^2-12x+36+y^2-4y+4+z^2+4z+4 \\ 14x-6y-10z &= 9 \end{aligned}$$

Thus, the set of points P is a plane in $\mathbb{R}^3$ with a normal vector $\mathbf{n} =< 7,-3,-5 >$. Also, notice that by this definition, this plane must go through the midpoint of A and B $\left(C\left(\frac{6-1}{2},\frac{5+2}{2},\frac{3-2}{2}\right)=\left(\frac{5}{2},\frac{7}{2},\frac{1}{2}\right)\right)$. Let us check that:

$$\begin{aligned} 14\left(\frac{5}{2}\right)-6\left(\frac{7}{2}\right)-10\left(\frac{1}{2}\right) &\stackrel{?}{=} 9 \\ 35-21-5 &\stackrel{?}{=} 9 \\ 9 &= 9 \end{aligned}$$

This shows us that the midpoint is indeed on the plane, and that the set of points P is the plane with normal $\mathbf{n}$ going through the point $C\left(\frac{5}{2},\frac{7}{2},\frac{1}{2}\right)$.

3.2 Dot Product and Cross Product

For the two vectors $\mathbf{A},\ \mathbf{B}$ in $\mathbb{R}^3$:

(a) Find the value of $(|\mathbf{A}\times\mathbf{B}|)^2+(\mathbf{A}\cdot\mathbf{B})^2$.

(b) Show that if $\mathbf{A}-\mathbf{B}$ and $\mathbf{A}+\mathbf{B}$ are orthogonal, then $\mathbf{A}$ and $\mathbf{B}$ must have the same length.

Solution

To solve part (a), we will write the expressions in terms of the magnitudes of $\mathbf{A}$ and $\mathbf{B}$:

$$(|\mathbf{A}\times\mathbf{B}|)^2+(\mathbf{A}\cdot\mathbf{B})^2=|\mathbf{A}|^2|\mathbf{B}|^2\cos^2\theta+|\mathbf{A}|^2|\mathbf{B}|^2\sin^2\theta$$
$$=|\mathbf{A}|^2|\mathbf{B}|^2(\cos^2\theta+\sin^2\theta)=|\mathbf{A}|^{\mathbf{2}}|\mathbf{B}|^{\mathbf{2}}\quad\text{(By the Pythagorean Identity)}$$
$$\implies \boxed{(|\mathbf{A}\times\mathbf{B}|)^2+(\mathbf{A}\cdot\mathbf{B})^2=|\mathbf{A}|^2|\mathbf{B}|^2}$$

Now, to deal with part (b), recall that two orthogonal vectors must have a vanishing dot product:

$$(\mathbf{A}-\mathbf{B})\cdot(\mathbf{A}+\mathbf{B})=0\implies \mathbf{A}\cdot\mathbf{A}+\mathbf{A}\cdot\mathbf{B}-\mathbf{B}\cdot\mathbf{A}-\mathbf{B}\cdot\mathbf{B}=0$$
$$\implies |\mathbf{A}|^2+\mathbf{A}\cdot\mathbf{B}-\mathbf{A}\cdot\mathbf{B}-|\mathbf{B}|^2=0\quad\text{(Since the dot product is commutative)}$$
$$\implies |\mathbf{A}|^2=|\mathbf{B}|^2\implies \boxed{|\mathbf{A}|=|\mathbf{B}|}$$

3.3 Lines in $\mathbb{R}^3$

Determine whether the following lines are parallel, intersect, or are skew. If they intersect, find the point of intersection.

$$l_1 : \frac{x-2}{1} = \frac{y-3}{-2} = \frac{z-1}{-3}$$
$$l_2 : \frac{x-3}{1} = \frac{y+4}{3} = \frac{z-2}{-7}$$

Solution

First off, since the direction vectors of each line (represented by numbers in the denominators) are not proportional we conclude that the two lines are not parallel. To find out whether or not the lines intersect, we will write the parametric equations of each line by setting each expression equal to a parameter:

$$\begin{aligned} &l_1 : t = \frac{x-2}{1} = \frac{y-3}{-2} = \frac{z-1}{-3} \\ &\implies x = t+2,\ y = -2t+3,\ z = -3t+1 \\ &l_2 : s = \frac{x-3}{1} = \frac{y+4}{3} = \frac{z-2}{-7} \\ &\implies x = s+3,\ y = 3s-4,\ z = -7s+2 \end{aligned}$$

Now, setting two of the coordinates in each line equal to one another (here, we choose x and y), we determine the values of the parameters:

$$\begin{aligned} &t+2 = s+3, \quad -2t+3 = 3s-4 \\ &2t+3s = 7, \quad t = 1+s \\ &\implies 2+2s+3s = 7 \implies s = 1 \implies t = 2 \end{aligned}$$

Now, we will evaluate the three coordinate and check if the lines agree at the values of the parameters that we just found:

$$\begin{aligned} &l_1 : x = 2+2 = 4,\ y = -2(2)+3 = -1,\ z = -3(2)+1 = -5 \\ &l_2 : x = 1+3 = 4,\ y = 3(1)-4 = -1,\ z = -7(1)+2 = -5 \\ &\implies \boxed{\text{Point of intersection is } (4,-1,-5)} \end{aligned}$$

Remark: had the two points at the specified parameter values not agreed, we would have determined that the lines are skew.

3.4 Distances Between Lines

Find the distance between the lines $\mathbf{r}_1 = (1+t)\mathbf{i} - 2\mathbf{j} - t\mathbf{k}$ and $\mathbf{r}_2 = -s\mathbf{i} + (2+s)\mathbf{j} - \mathbf{k}$.

Solution

Rewriting these line equations in the form $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$, we see that $\mathbf{r_1} = \mathbf{i} - 2\mathbf{j} + (\mathbf{i} - \mathbf{k})t$ and $\mathbf{r_2} = 2\mathbf{j} - \mathbf{k} + (-\mathbf{i} + \mathbf{j})t$. This implies that the direction vector for $\mathbf{r_1}$ is $\mathbf{A} = \langle 1, 0, -1 \rangle$, for $\mathbf{r_2}$ is $\mathbf{B} = \langle -1, 1, 0 \rangle$.

We will proceed by finding a vector normal to both lines by taking the cross product of both direction vectors:

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{vmatrix} = \langle 1, 1, 1 \rangle$$

Finally, we will find a plane with normal vector $\langle 1, 1, 1 \rangle$ containing one of the lines and parallel to the other, and thus reduce the problem to finding the distance between a point and a plane. The equation of a general plane is $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$ where $\langle a, b, c \rangle$ represents the normal vector to the plane and (x_0, y_0, z_0) a point on the plane. If we choose for the plane to contain $\mathbf{r_1}$ then the equation must be satisfied by a point on that line, say $P(1, -2, 0)$ which we get by simply setting $t = 0$:

$$(x-1) + (y+2) + (z) = 0 \implies x + y + z + 1 = 0$$

Now, to find the distance to the point $Q(0, 2, -1)$ (again, found by setting $s = 0$) we will use the formula for the distance between a point and a plane:

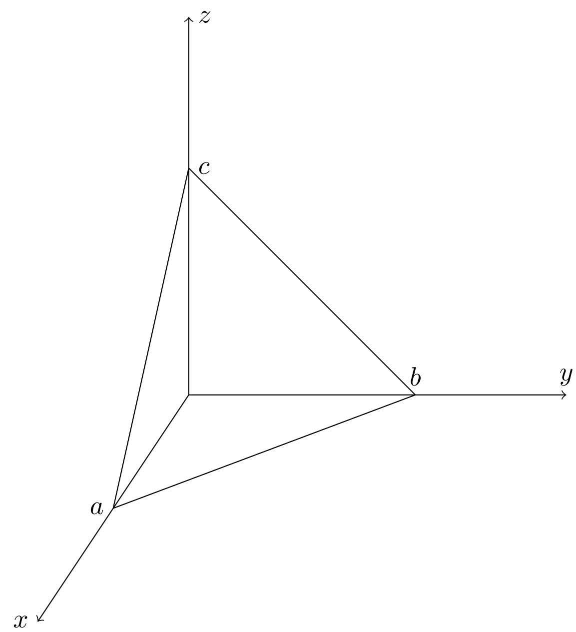
$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}} = \frac{|0 + 2 - 1 + 1|}{\sqrt{1^2 + 1^2 + 1^2}} = \boxed{\frac{2}{\sqrt{3}}}$$

3.5 Equations of Planes

Find the equation of the plane with x-intercept a, y-intercept b, and z-intercept c.

Solution

Here, drawing a picture as shown will help us visualize the problem:



Notice that the sides of the tetrahedron drawn in the first octant can be written as vectors by taking the differences of the intercepts, namely:

$$\mathbf{A} = (a, 0, 0) - (0, 0, c) = \langle a, 0, -c\rangle, \quad \mathbf{B} = (0, b, 0) - (0, 0, c) = \langle 0, b, -c\rangle$$

Now, notice that since these two vectors are contained in the plane, we can take the cross product of those two vectors to determine the normal to the plane:

$$\mathbf{n} = \mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a & 0 & -c \\ 0 & b & -c \end{vmatrix} = \langle bc, ac, ab\rangle$$

From the general equation of a plane and using the x-intercept as a point on the plane, we have the equation of the plane to be (after simplification)

$$\boxed{\frac{1}{a}x + \frac{1}{b}y + \frac{1}{c}z = 1}$$

3.6 More Equations of Planes

Find the equation of the plane that contains the line of intersection of the planes $x - z = 1$ and $y + 2z = 3$ and is perpendicular to the plane $x + y - 2z = 1$.

Solution

To begin approaching this problem, we must first find the direction vector of the line of intersection of the two planes. To do so, notice that the line of intersection is contained in both planes, meaning it is perpendicular to both the normal vectors. Thus, we can recover the direction vector by taking the cross product of the two normal vectors (labeled $\mathbf{n_1}$ and $\mathbf{n_2}$)

$$\mathbf{n_1} = \langle 1, 0, -1\rangle, \quad \mathbf{n_1} = \langle 0, 1, 2\rangle$$

$$\mathbf{n_1} \times \mathbf{n_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 0 & 1 & 2 \end{vmatrix} = \langle 1, -2, 1\rangle$$

Now that we have the direction vector for the line of intersection, we must use that to determine the normal vector to the plane. To do so we note that since the plane we want is perpendicular to $x + y - 2z = 1$ it must contain the normal vector $\langle 1, 1, -2\rangle$. Also, since it contains the vector $\langle 1, -2, 1\rangle$ we can use another cross product to determine the normal vector to the plane we are interested in:

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{vmatrix} = \langle 3, 3, 3\rangle$$

Finally, to completely determine the plane, we must find a point contained in the plane. To do so, we will find a point on the line of intersection of $x - z = 1$ and $y + 2z = 3$ by setting $z = 0$ (you can choose to set any of the three variables to any number) and solving for the other two:

$$\begin{aligned}
&\text{Set } z = 0 \implies x = 1 \implies y = 3 \\
&\implies P(1, 3, 0) \text{ belongs to the plane of interest.} \\
&a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \text{ (general equation of a plane)} \\
&3(x - 1) + 3(y - 3) + 3z = 0 \implies 3x + 3y + 3z = 12 \\
&\implies \boxed{x + y + z = 4}
\end{aligned}$$

3.7 Parametric Equations of Curves

Two particles travel along the following space curves:

$$\mathbf{r}_1(t) = \langle t, t^2, t^3 \rangle, \quad \mathbf{r}_2(t) = \langle 1+2t, 1+6t, 1+14t \rangle$$

Do the particles collide? Do their paths cross?

Solution

To find out if the two particles collide, we will set two of the coordinates equal to one another and solve for t:

$$t = 1+2t \implies -t = 1 \implies t = -1$$

Now, for $t = -1$, we will check whether or not both particles will occupy the same point:

$$\mathbf{r}_1(-1) = \langle -1, 1, -1 \rangle, \quad \mathbf{r}_2(-1) = \langle -1, -5, -13 \rangle$$
$$\implies \boxed{\text{particles do not collide}}$$

Now, if their paths were to intersect, it means that the particles occupy the same points at different parameter values. Thus, we will consider a different parameter for $\mathbf{r}_2$ (say, s) and attempt to solve for a point of intersection:

$\mathbf{r}_1(t) = \langle t, t^2, t^3 \rangle, \quad \mathbf{r}_2(t) = \langle 1+2s, 1+6s, 1+14s \rangle$

At intersection, $\quad t = 1+2s, \quad t^2 = 1+6s, \quad t^3 = 1+14s$

$\implies (1+2s)^2 = 1+6s \quad$ (Using the first two equations)

$\implies 4s^2 + 4s + 1 = 1 + 6s \implies 2s(2s-1) = 0$

$\implies s = 0$ or $s = \dfrac{1}{2}$

$\implies t = 1$ or $t = 2$

Here, we must check that the z coordinate of both particles agree at the parameter values above, which they do

$$\boxed{\text{Thus, intersection of path occurs at } \mathbf{r}_1(1) = \langle 1,1,1 \rangle,\ \mathbf{r}_1(2) = \langle 2,4,8 \rangle}$$

Remark: for the last step, we could have used the s values and plugged those into $\mathbf{r}_2$ to arrive at the same answer.

3.8 Sketching Quadric Surfaces

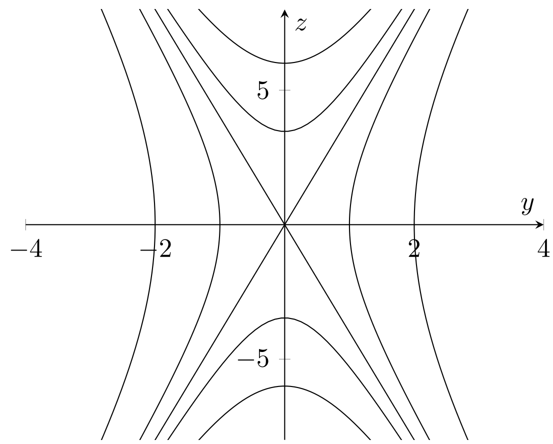
Use traces to sketch and identify the surface with the following equation:

$$4x^2 - 16y^2 + z^2 = 16$$

Solution

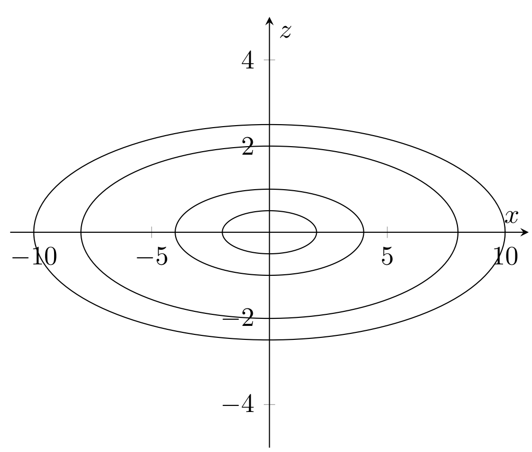
As the problem suggests, we will use traces, so here we go:

$$\begin{aligned}&x\text{-traces: } (x = k)\\ &4k^2 - 16y^2 + z^2 = 16\\ &\implies \frac{z^2}{16-4k^2} - \frac{16y^2}{16-4k^2} = 1\end{aligned}$$



For $k > 2$ we will simply get more horizontal hyperbolas in the xz plane.

$$\begin{aligned}&y\text{-traces: } (y = k)\\ &4x^2 - 16k^2 + z^2 = 16\\ &\implies \frac{x^2}{16+16k^2} + \frac{16z^2}{16+16k^2} = 1\end{aligned}$$

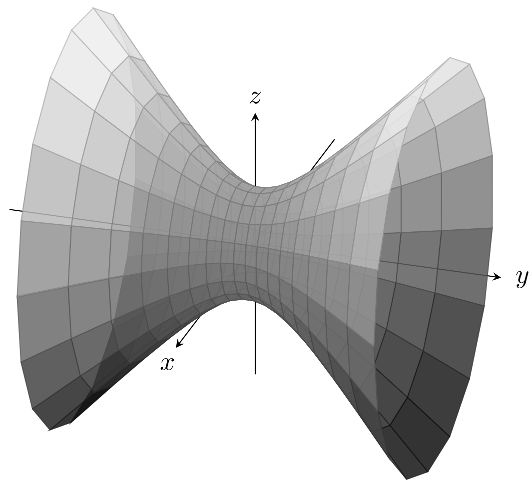


Here, the smallest possible ellipse is the one generated by the $k = 0$ case since positive and negative k values have the same effects.

Sketching Quadric Surfaces (Continued)

$$\begin{aligned}&z\text{-traces: } (z = k)\\&4x^2 - 16y^2 + k^2 = 16\\&\implies \frac{4x^2}{16-k^2} - \frac{16y^2}{16-k^2} = 1\end{aligned}$$

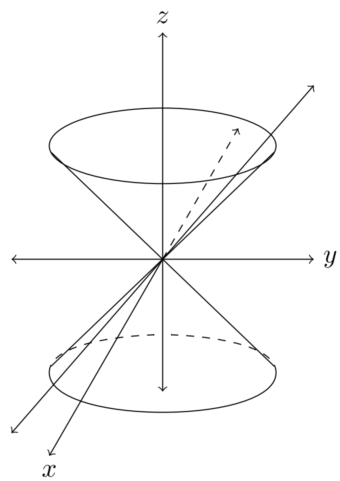
It is pretty clear that these traces will also be hyperbolas (just like the x-traces). Thus, we can conclude from the pictures that the surface at hand is hyperboloid of one sheet.



3.9 Surfaces of Rotations

Given the line $z = y \tan \theta$ in the yz-plane, where $0 < \theta < \pi/2$, find the equation of the surface generated by rotating the line about the z-axis.

Solution



Revolving a line about the z-axis will generate a cone, hence any plane $z = k$ will intersect the cone in a circle.

$$x^2 + y^2 = r^2$$

It only remains to find the radius of the circle in terms of z. The easiest way to do this is let $x = 0$, $z = k$, and solve for y (i.e. the radius of the circle) from the given line equation.

$$r = y = \frac{k}{\tan \theta}$$

As z varies you get all the radii; $r = \dfrac{z}{\tan \theta}$. So the equation of the cone is simply

$$x^2 + y^2 = \frac{z^2}{\tan^2 \theta}$$

As a quick check let $\theta = \pi/4$, now we expect the standard cone: $x^2 + y^2 = z^2$, plugging in for θ above does indeed yield this.

3.10 Equations of Quadric Surfaces

Find an equation for the surface consisting of all points P for which the distance from P to the x-axis is twice the distance from P to the yz-plane. Identify the surface.

Solution

We consider a general point $P(x, y, z)$ and think about how we would represent these distances. A quick sketch of $\mathbb{R}^3$ shows us that the distance from the yz-plane to the point is simply the x coordinate. Furthermore, the shortest distance between any point and the x-axis is along the line perpendicular to the x-axis going through the point P. This means that for a point P:

$$d(P, x\text{-axis}) = d(P, (x, 0, 0)) = \sqrt{(x-x)^2 + (y-0)^2 + (z-0)^2} = \sqrt{y^2 + z^2}$$

Thus,

$$\begin{aligned} d(P, x\text{-axis}) &= 2d(P, yz\text{-plane}) \\ \sqrt{y^2 + z^2} &= 2x \\ \boxed{y^2 + z^2 = 4x^2} \end{aligned}$$

We can easily identify the equation as the equation of a cone (one can also use traces to identify this surface)

3.11 Contours

Draw a contour map of the following function showing several level curves:

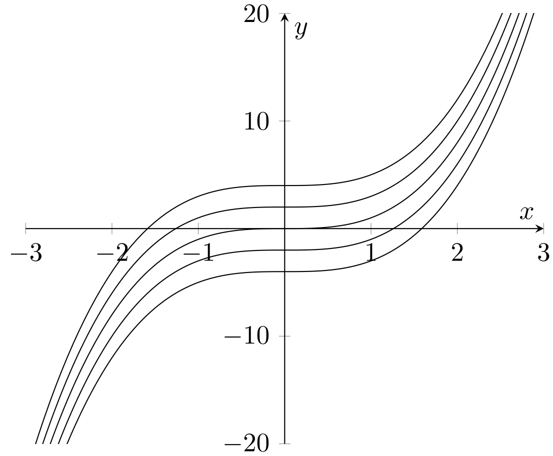
$$f(x, y) = x^3 - y$$

Solution

Contour maps are obtained by drawing a collection of level curves defined by $f(x, y) = c$ for some constant c. In our case, level curves are very simple to graph:

$$c = x^3 - y \implies y = x^3 - c$$

This equation simply describes a cubic shifted vertically. Below is the contour map showing a few level curve.



3.12 Points in Cylindrical Coordinates

Change the following points to the specified coordinate system:

(a) $(2\sqrt{3}, 2, -1)$ from rectangular coordinates to cylindrical coordinates

(b) $(1, 1, 1)$ from cylindrical coordinates to rectangular coordinates

Solution

First, let us establish the relationships we will use the for both parts:

$$x = r\cos\theta, \quad y = r\sin\theta, \quad 0 \le \theta < 2\pi$$
$$\implies x^2 + y^2 = r^2, \quad \frac{y}{x} = \tan\theta$$

Now, for part (a) we have our point in (x, y, z) form and we need to convert to (r, θ, z), and that we will:

$$r = \sqrt{x^2 + y^2} = \sqrt{12 + 4} = \sqrt{16} = 4$$
$$\theta = \arctan\frac{y}{x} = \arctan\frac{2}{2\sqrt{3}} = \frac{\pi}{6}$$

Thus, the cylindrical representation of the point is $\boxed{\left(4, \frac{\pi}{6}, -1\right)}$

Part (b) will be approached in a similar manner, just backwards:

$$x = r\cos\theta = (1)\cos(1) = \cos 1 \approx 0.54030$$
$$y = r\sin\theta = (1)\sin(1) = \sin 1 \approx 0.84147$$

Thus, the rectangular representation of the point is $\boxed{(\cos 1, \sin 1, 1) \approx (0.54030, 0.84147, 1)}$

3.13 Points in Spherical Coordinates

Change the following points to the specified coordinate system:

(a) $(4, -\frac{\pi}{4}, \frac{\pi}{3})$ from spherical coordinates to rectangular coordinates

(b) $(-1, 1, \sqrt{2})$ from rectangular coordinates to spherical coordinates

Solution

Once more, we will establish the relationships we will use for both parts:

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho\cos\phi$$
$$r = \rho\sin\phi, \quad x^2 + y^2 = r^2, \quad \rho^2 = x^2 + y^2 + z^2$$

For part (a) we are given a point in the form (ρ, θ, ϕ) and so:

$$x = \rho \sin\phi \cos\theta = (4) \sin\left(\frac{\pi}{3}\right) \cos\left(\frac{-\pi}{4}\right) = 4\left(\frac{\sqrt{3}}{2}\right)\frac{\sqrt{2}}{2} = \sqrt{6}$$
$$y = \rho \sin\phi \sin\theta = (4) \sin\left(\frac{\pi}{3}\right) \sin\left(\frac{-\pi}{4}\right) = 4\left(\frac{\sqrt{3}}{2}\right)\frac{-\sqrt{2}}{2} = -\sqrt{6}$$
$$z = \rho\cos\phi = (4)\cos\left(\frac{\pi}{3}\right) = 4\left(\frac{1}{2}\right) = 2$$

Thus, the rectangular coordinates of the point are $\boxed{\left(\sqrt{6},\ -\sqrt{6},\ 2\right)}$

For part (b):

$$\rho^2 = x^2 + y^2 + z^2 = (-1)^2 + (1)^2 + (\sqrt{2})^2 = 4$$
$$\implies \rho = 2$$
$$z = \rho\cos\phi \implies \cos\phi = \frac{z}{\rho} = \frac{\sqrt{2}}{2}$$
$$\text{Since } 0 \le \phi \le \pi \text{ and } \cos\phi = \frac{\sqrt{2}}{2} \implies \phi = \frac{\pi}{4}$$

Here, we will take an intermediate step and calculate $r^2 = (-1)^2 + (1)^2 = 2$

$$x = \rho\sin\phi\cos\theta = r\cos\theta$$
$$\implies \cos\theta = \frac{-1}{\sqrt{2}} = \frac{-\sqrt{2}}{2}$$
$$\implies \theta = \frac{3\pi}{4} \text{ or } \theta = \frac{5\pi}{4}$$
$$y = \rho\sin\phi\sin\theta = r\sin\theta$$
$$\implies \sin\theta = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$
$$\implies \theta = \frac{\pi}{4} \text{ or } \theta = \frac{3\pi}{4}$$
$$\text{Thus } \theta = \frac{3\pi}{4}$$

Finally, we can write the spherical coordinates as $\boxed{\left(2,\ \frac{3\pi}{4},\ \frac{\pi}{4}\right)}$

3.14 Parametric Equations of Surfaces

Find a parametric representation for the following surfaces:

(a) The part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the cone $z = \sqrt{x^2 + y^2}$

(b) The part of the ellipsoid $x^2 + 2y^2 + 3z^2 = 1$ that lies to the left of the xz-plane

Solution

To tackle part (a) we will use the spherical coordinate system. Recall that:

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi$$
$$r = \rho \sin\phi, \quad x^2 + y^2 = r^2, \quad \rho^2 = x^2 + y^2 + z^2$$

The last equation in the list implies that $\rho^2 = 4 \implies \rho = 2$. It is easy to see (a sketch might help you here) that $0 \leq \theta < 2\pi$. Now, to specify the variation of ϕ, we look at the equation of the cone:

$$z = z = \sqrt{x^2 + y^2} \implies \rho\cos\phi = \rho\sin\phi \implies \cos\phi = \sin\phi$$
$$\text{Since } 0 \leq \phi \leq \pi \implies \phi = \frac{\pi}{4} \text{ for the cone surface}$$

Since we want to include all of the sphere's surface within the cone, we say that $0 \leq \phi \leq \frac{\pi}{4}$ and thus, our parametric equations look as follows:

$$\boxed{x = 2\cos\theta\sin\phi, \quad y = 2\sin\theta\sin\phi, \quad z = 2\cos\phi}$$
$$0 \leq \phi \leq \frac{\pi}{4}, \quad 0 \leq \theta < 2\pi \text{ (You must include these inequalities for a complete solution)}$$

As for part (b), since being to the left of the xz-plane implies that the values of y are all negative, we will solve for y in terms of x and z and consider only the negative values of y:

$$x^2 + 2y^2 + 3z^2 = 1 \implies y = -\sqrt{\frac{1}{2}(1 - x^2 - 3z^2)}$$
$$\implies \boxed{x = u, \quad y = -\sqrt{\frac{1}{2}(1 - u^2 - 3v^2)}, \quad z = v}$$

4 Linear Algebra

4.1 Systems of Equations

Solve the following system of equation using substitution and/or elimination:

$$\begin{aligned} 2x - y + z &= 3 \\ 3x + 2y - z &= -1 \\ x - 3y + 2z &= 2 \end{aligned}$$

Solution

We will begin by adding the first two equations to eliminate z:

$$\begin{array}{cccc} 2x & -y & +z & = 3 \\ 3x & +2y & -z & = -1 \\ \hline 5x & +y & & = 2 \end{array}$$

Similarly, we will use the second and third equations by first multiplying the first equation by 2 and then adding:

$$\begin{array}{cccc} 6x & +4y & -2z & = -2 \\ x & -3y & +2z & = 2 \\ \hline 7x & +y & & = 0 \end{array}$$

Now, we will solve the following system using substitution:

$5x + y = 2$

$7x + y = 0 \implies y = -7x$

Plugging that into the first equation: $5x - 7x = 2$

$\implies -2x = 2 \implies \boxed{x = -1 \implies y = -7(-1) = 7}$

Finally, using any of the 3 original equations (here we use the first):

$\boxed{z = 3 - 2(-1) + 7 = 12}$

4.2 More on Systems of Equations

Solve the system of equations from the previous problem using Gaussian elimination. Interpret your solution.

Solution

We start by writing the equations in matrix form and reducing:

$$\left(\begin{array}{ccc|c} 2 & -1 & 1 & 3 \\ 3 & 2 & -1 & -1 \\ 1 & -3 & 2 & 2 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & -3 & 2 & 2 \\ 3 & 2 & -1 & -1 \\ 2 & -1 & 1 & 3 \end{array}\right) \xrightarrow{-3R_1 + R_2} \left(\begin{array}{ccc|c} 1 & -3 & 2 & 2 \\ 0 & 11 & -7 & -7 \\ 2 & -1 & 1 & 3 \end{array}\right)$$

$$\xrightarrow{-2R_1 + R_3} \left(\begin{array}{ccc|c} 1 & -3 & 2 & 2 \\ 0 & 11 & -7 & -7 \\ 0 & 5 & -3 & -1 \end{array}\right) \xrightarrow{R_2 \to \frac{1}{11}R_2} \left(\begin{array}{ccc|c} 1 & -3 & 2 & 2 \\ 0 & 1 & -\frac{7}{11} & -\frac{7}{11} \\ 0 & 5 & -3 & -1 \end{array}\right) \xrightarrow{3R_2 + R_1} \left(\begin{array}{ccc|c} 1 & 0 & \frac{1}{11} & \frac{1}{11} \\ 0 & 1 & -\frac{7}{11} & -\frac{7}{11} \\ 0 & 5 & -3 & -1 \end{array}\right)$$

$$\xrightarrow{-5R_2 + R_3} \left(\begin{array}{ccc|c} 1 & 0 & \frac{1}{11} & \frac{1}{11} \\ 0 & 1 & -\frac{7}{11} & -\frac{7}{11} \\ 0 & 0 & \frac{2}{11} & \frac{24}{11} \end{array}\right) \xrightarrow{R_3 \to \frac{11}{2}R_3} \left(\begin{array}{ccc|c} 1 & 0 & \frac{1}{11} & \frac{1}{11} \\ 0 & 1 & -\frac{7}{11} & -\frac{7}{11} \\ 0 & 0 & 1 & 12 \end{array}\right) \xrightarrow{\frac{7}{11}R_3 + R_2} \left(\begin{array}{ccc|c} 1 & 0 & \frac{1}{11} & \frac{1}{11} \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 12 \end{array}\right)$$

$$\xrightarrow{-\frac{1}{11}R_3 + R_1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 12 \end{array}\right)$$

Thus,

$$\boxed{\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 7 \\ 12 \end{pmatrix}}$$

Note that we arrived to the same results as in the previous problem, as is expected. The results tell us that all three planes we were given the equations of intersect at a single point and the point is $(x,\ y,\ z) = (-1,\ 7,\ 12)$

4.3 Reduced Row-Echelon

Given that a system of 3 linear equations, each of 3 variables, has the following solution, construct the reduced row-echelon matrix the solution came from.

$$x = 1 - 4t, \quad y = 3 + 3t, \quad z = 2 - t$$

Solution

To express the solution in reduced row-echelon, we need to reparametrize our solution to where z is a parameter:

$$z = 2 - t = s \implies t = 2 - s$$
$$x = 1 - 4t = 1 - 4(2 - s) = -7 + 4s, \quad y = 3 + 3(2 - s) = 9 - 3s$$

Now we can write our matrix as follows:

$$\boxed{\left(\begin{array}{ccc|c} 1 & 0 & -4 & -7 \\ 0 & 1 & 3 & 9 \\ 0 & 0 & 0 & 0 \end{array}\right)}$$

4.4 Determinants

Calculate the determinant of the following matrix.

$$\begin{pmatrix} 3 & 1 & 3 & 6 \\ 4 & -7 & -3 & 5 \\ 1 & 3 & 4 & -3 \\ 3 & 0 & 2 & 7 \end{pmatrix}$$

Solution

An efficient way of calculating large determinant is to row-reduce the matrix until a column has all zeros except for one entry. We start by making the first column all zeros except for the a_{31} entry.

$$\begin{pmatrix} 3 & 1 & 3 & 6 \\ 4 & -7 & -3 & 5 \\ 1 & 3 & 4 & -3 \\ 3 & 0 & 2 & 7 \end{pmatrix} \xrightarrow{-3R_3+R_4} \begin{pmatrix} 3 & 1 & 3 & 6 \\ 4 & -7 & -3 & 5 \\ 1 & 3 & 4 & -3 \\ 0 & -9 & -10 & 16 \end{pmatrix} \xrightarrow{-4R_3+R_2}$$
$$\begin{pmatrix} 3 & 1 & 3 & 6 \\ 0 & -19 & -19 & 17 \\ 1 & 3 & 4 & -3 \\ 0 & -9 & -10 & 16 \end{pmatrix} \xrightarrow{-3R_3+R_1} \begin{pmatrix} 0 & -8 & -9 & 15 \\ 0 & -19 & -19 & 17 \\ 1 & 3 & 4 & -3 \\ 0 & -9 & -10 & 16 \end{pmatrix}$$

Now do the regular cofactor expansion for the determinant:

$$\begin{vmatrix} 0 & -8 & -9 & 15 \\ 0 & -19 & -19 & 17 \\ 1 & 3 & 4 & -3 \\ 0 & -9 & -10 & 16 \end{vmatrix} = 1 \begin{vmatrix} -8 & -9 & 15 \\ -19 & -19 & 17 \\ -9 & -10 & 16 \end{vmatrix} = -8 \begin{vmatrix} -19 & 17 \\ -10 & 16 \end{vmatrix} + 9 \begin{vmatrix} -19 & 17 \\ -9 & 16 \end{vmatrix} + 15 \begin{vmatrix} -19 & -19 \\ -9 & -10 \end{vmatrix}$$
$$= -8(-304 + 170) + 9(-304 + 153) + 15(190 - 171)$$
$$= 1072 - 1359 + 285$$
$$= -2$$

4.5 Inverses and Transpositions

A matrix A is called orthogonal if $AA^T = I$, where I is the identity matrix (i.e. $A^{-1} = A^T$). Given the following matrix, compute the inverse using (a) Row reduction, and (b) using 2×2 inverse formula. Verify that A is in fact orthogonal.

$$A = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Solution

We begin finding the inverse in part (a) by adjoining the identity matrix and reducing as follows:

$$\left(\begin{array}{cc|cc} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 1 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|cc} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 1 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 & 0 \end{array}\right) \xrightarrow{R_1 + R_2} \left(\begin{array}{cc|cc} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 1 \\ 0 & \frac{2}{\sqrt{2}} & 1 & 1 \end{array}\right)$$

$$\xrightarrow{R_2 \to \frac{1}{\sqrt{2}} R_2} \left(\begin{array}{cc|cc} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 1 \\ 0 & 1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{array}\right) \xrightarrow{-\frac{1}{\sqrt{2}} R_2 + R_1} \left(\begin{array}{cc|cc} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{array}\right) \xrightarrow{R_1 \to \sqrt{2} R_1} \left(\begin{array}{cc|cc} 1 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{array}\right)$$

$$\implies \boxed{A^{-1} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}}$$

Next we will compute the inverse using the following formula:

$$\text{For } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$\implies A^{-1} = \frac{1}{-\frac{1}{2} - \frac{1}{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Finally, to verify that the matrix A is orthogonal, notice that this matrix is very special since $A^T = A = A^{-1}$:

$$AA^T = AA^{-1} = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

4.6 Products of Determinants

For any two 2×2 matrices A and B, show that $\det(AB) = \det(A)\det(B)$.

Solution

Consider

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad B = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$
$$\implies \det(A) = ad - bc, \quad \det(B) = a'd' - b'c'$$

We can compute the right hand side of the equation as follows:

$$\det(A)\det(B) = (ad - bc)(a'd' - b'c') = ada'd' - adb'c' - bca'd' + bcb'c'$$

Now to compute the left hand side, we first need to multiply the matrices:

$$AB = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{pmatrix}$$
$$\det(AB) = (aa' + bc')(cb' + dd') - (ca' + dc')(ab' + bd')$$
$$= \cancel{aca'b'} + ada'd' + bcb'c' + \cancel{bdc'd'} - \cancel{aca'b'} - bca'd' - adb'c' - \cancel{bdc'd'}$$
$$= ada'd' + bcb'c' - bca'd' - adb'c'$$

As we can see, the left and right hand sides agree! Notice that a nice application of this property is when considering a matrix and its own inverse:

$$\det(I) = \det(AA^{-1}) = \det(A)\det(A^{-1})$$
$$\implies 1 = \det(A)\det(A^{-1}) \implies \det(A^{-1}) = \frac{1}{\det(A)}$$

4.7 Matrix Arithmetic and Characteristic Equations

The Caley-Hamilton theorem states that "A matrix satisfies its own characteristic equation." Verify this theorem for the following matrix

$$M = \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}$$

Solution

First find the characteristic equation

$$\begin{vmatrix} 5-\lambda & -2 \\ -2 & 2-\lambda \end{vmatrix} = 0$$
$$(5-\lambda)(2-\lambda) - 4 = 0$$
$$\lambda^2 - 7\lambda + 6 = 0$$

Now plug M in for λ,

$$\begin{aligned}
&\begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}^2 - 7\begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix} + 6\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}\begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix} - \begin{pmatrix} 35 & -14 \\ -14 & 14 \end{pmatrix} + \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix} \\
&= \begin{pmatrix} 29 & -14 \\ -14 & 8 \end{pmatrix} - \begin{pmatrix} 35 & -14 \\ -14 & 14 \end{pmatrix} + \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix} \\
&= \begin{pmatrix} 29-35+6 & -14+14+0 \\ -14+14+0 & 8+14+6 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\end{aligned}$$

Hence M satisfies its own characteristic equation.

4.8 Eigenvalues and Eigenvectors

Find the eigenvalues of the real symmetric matrix

$$\begin{pmatrix} A & H \\ H & B \end{pmatrix}$$

Where $A, H, B \in \mathbb{R}$. Show that the eigenvalues are real. In the case where $A = 4$, $B = 1$, and $H = 2$, show that the eigenvectors are orthogonal.

Solution

$$\begin{aligned}
\begin{pmatrix} A-\lambda & H \\ H & B-\lambda \end{pmatrix} &= 0 \\
(A-\lambda)(B-\lambda) - H^2 &= 0 \\
\lambda^2 - \lambda(A+B) + AB - H^2 &= 0 \\
\lambda &= \frac{A+B \pm \sqrt{(A+B)^2 - 4(AB - H^2)}}{2} \\
&= \frac{A+B \pm \sqrt{A^2 + B^2 + 2AB - 4AB + 4H^2}}{2} \\
&= \frac{A+B \pm \sqrt{(A-B)^2 + 4H^2}}{2} \qquad (1)
\end{aligned}$$

Equation (1) gives both eigenvalues. Notice that since argument in the square root is a sum of squares it is positive and hence both eigenvalues are real.
Now find the eigenvalues and eigenvectors of

$$\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}$$

Using equation (1) the eigenvalues are

$$\begin{aligned}
\lambda_+ &= \frac{5 + \sqrt{9+16}}{2} = \frac{10}{2} = 5 \\
\lambda_- &= \frac{5 - \sqrt{9+16}}{2} = \frac{5-5}{2} = 0
\end{aligned}$$

Find the eigenvector corresponding to λ_+

$$\begin{aligned}
\begin{pmatrix} 4-5 & 2 \\ 2 & 1-5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\
-x + 2y &= 0 \\
x &= 2y
\end{aligned}$$

Choose $x = 1$, so then $y = 2$ and the eigenvector is

$$\mathbf{v}_+ = \begin{pmatrix} 1 \\ 1/2 \end{pmatrix}$$

Find the eigenvector corresponding to λ_-

$$\begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$4x + 2y = 0$$

Choose $x = 1$, then $y = -2$ and the eigenvector is

$$\mathbf{v}_- = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Take the dot product of the two eigenvectors to show that they are orthogonal

$$\mathbf{v}_+ \cdot \mathbf{v}_- = \begin{pmatrix} 1 \\ 1/2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \end{pmatrix} = 1 - 1 = 0$$

Therefore the eigenvectors are orthogonal.