Contents

1 Geometry of $\mathbb{R}^2$

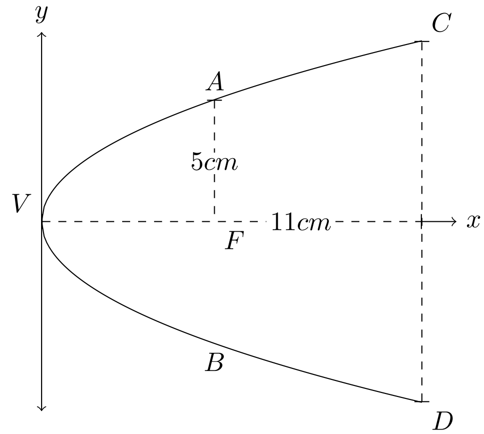
1.1 Conic Sections

A cross-section of a parabolic reflector is shown in the figure. The bulb is located at the focus and the opening at the focus is 10 cm.

(a) Find an equation of the parabola.

(b) Find the diameter of the opening $|CD|$, 11 cm from the vertex.

(c) Parametrize the bottom half of the parabola.



1.2 Parametric Equations

By first eliminating the parameter, describe the motion of a particle following the graph of the parametric equations:

$$x = 2\sin t, \quad y = 4 + \cos t, \quad 0 \leq t \leq \frac{3\pi}{2}$$

1.3 More Parametric Equations

Find the Cartesian equations for each of the curves described by the parametric equations below:

(a) $x = e^t - 1, \quad y = e^{2t}$

(b) $x = \tan^2\theta, \quad y = \sec\theta, \quad \frac{-\pi}{2} < \theta < \frac{\pi}{2}$

1.4 Polar Coordinates

Sketch the curve

$$(x^2 + y^2)^3 = 4x^2y^2$$

2 Complex Numbers

2.1 Polar Form

Use Euler's formula to prove that:

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}, \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

2.2 Roots of Unity

Find all the cube roots of 1 and show that if one of the complex roots (i.e. nonzero imaginary part) is labeled z then the other complex root is z^2.

3 Geometry of $\mathbb{R}^3$

3.1 The Distance Formula

Describe the set of points P such that the distance from P to $A(-1,5,3)$ is equal to the distance from P to $B(6,2,-2)$.

3.2 Dot Product and Cross Product

For the two vectors $\mathbf{A},\ \mathbf{B}$ in $\mathbb{R}^3$:

(a) Find the value of $(|\mathbf{A}\times\mathbf{B}|)^2 + (\mathbf{A}\cdot\mathbf{B})^2$.

(b) Show that if $\mathbf{A}-\mathbf{B}$ and $\mathbf{A}+\mathbf{B}$ are orthogonal, then $\mathbf{A}$ and $\mathbf{B}$ must have the same length.

3.3 Lines in $\mathbb{R}^3$

Determine whether the following lines are parallel, intersect, or are skew. If they intersect, find the point of intersection.

$$l_1 : \frac{x-2}{1} = \frac{y-3}{-2} = \frac{z-1}{-3}$$
$$l_2 : \frac{x-3}{1} = \frac{y+4}{3} = \frac{z-2}{-7}$$

3.4 Distances Between Lines

Find the distance between the lines $\mathbf{r}_1 = (1+t)\mathbf{i} - 2\mathbf{j} - t\mathbf{k}$ and $\mathbf{r}_2 = -s\mathbf{i} + (2+s)\mathbf{j} - \mathbf{k}$.

3.5 Equations of Planes

Find the equation of the plane with x-intercept a, y-intercept b, and z-intercept c.

3.6 More Equations of Planes

Find the equation of the plane that contains the line of intersection of the planes $x-z=1$ and $y+2z=3$ and is perpendicular to the plane $x+y-2z=1$.

3.7 Parametric Equations of Curves

Two particles travel along the following space curves:

$$\mathbf{r}_1(t) = \langle t, t^2, t^3\rangle, \quad \mathbf{r}_2(t) = \langle 1+2t, 1+6t, 1+14t\rangle$$

Do the particles collide? Do their paths cross?

3.8 Sketching Quadric Surfaces

Use traces to sketch and identify the surface with the following equation:

$$4x^2 - 16y^2 + z^2 = 16$$

3.9 Surfaces of Rotations

Given the line $z = y\tan\theta$ in the yz-plane, where $0 < \theta < \pi/2$, find the equation of the surface generated by rotating the line about the z-axis.

3.10 Equations of Quadric Surfaces

Find an equation for the surface consisting of all points P for which the distance from P to the x-axis is twice the distance from P to the yz-plane. Identify the surface.

3.11 Contours

Draw a contour map of the following function showing several level curves:

$$f(x, y) = x^3 - y$$

3.12 Points in Cylindrical Coordinates

Change the following points to the specified coordinate system:

(a) $(2\sqrt{3}, 2, -1)$ from rectangular coordinates to cylindrical coordinates

(b) $(1, 1, 1)$ from cylindrical coordinates to rectangular coordinates

3.13 Points in Spherical Coordinates

Change the following points to the specified coordinate system:

(a) $(4, -\frac{\pi}{4}, \frac{\pi}{3})$ from spherical coordinates to rectangular coordinates

(b) $(-1, 1, \sqrt{2})$ from rectangular coordinates to spherical coordinates

3.14 Parametric Equations of Surfaces

Find a parametric representation for the following surfaces:

(a) The part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the cone $z = \sqrt{x^2 + y^2}$

(b) The part of the ellipsoid $x^2 + 2y^2 + 3z^2 = 1$ that lies to the left of the xz-plane

4 Linear Algebra

4.1 Systems of Equations

Solve the following system of equation using substitution and/or elimination:

$$\begin{aligned} 2x - y + z &= 3 \\ 3x + 2y - z &= -1 \\ x - 3y + 2z &= 2 \end{aligned}$$

4.2 More on Systems of Equations

Solve the system of equations from the previous problem using Gaussian elimination. Interpret your solution.

4.3 Reduced Row-Echelon

Given that a system of 3 linear equations, each of 3 variables, has the following solution, construct the reduced row-echelon matrix the solution came from.

$$x = 1 - 4t, \quad y = 3 + 3t, \quad z = 2 - t$$

4.4 Determinants

Calculate the determinant of the following matrix.

$$\begin{pmatrix} 3 & 1 & 3 & 6 \\ 4 & -7 & -3 & 5 \\ 1 & 3 & 4 & -3 \\ 3 & 0 & 2 & 7 \end{pmatrix}$$

4.5 Inverses and Transpositions

A matrix A is called orthogonal if $AA^T = I$, where I is the identity matrix (i.e. $A^{-1} = A^T$). Given the following matrix, compute the inverse using (a) Row reduction, and (b) using 2×2 inverse formula. Verify that A is in fact orthogonal.

$$A = \begin{pmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

4.6 Products of Determinants

For any two 2×2 matrices A and B, show that $\det(AB) = \det(A)\det(B)$.

4.7 Matrix Arithmetic and Characteristic Equations

The Caley-Hamilton theorem states that "A matrix satisfies its own characteristic equation." Verify this theorem for the following matrix

$$M = \begin{pmatrix} 5 & -2 \\ -2 & 2 \end{pmatrix}$$

4.8 Eigenvalues and Eigenvectors

Find the eigenvalues of the real symmetric matrix

$$\begin{pmatrix} A & H \\ H & B \end{pmatrix}$$

Where $A, H, B \in \mathbb{R}$. Show that the eigenvalues are real. In the case where $A = 4$, $B = 1$, and $H = 2$, show that the eigenvectors are orthogonal.