Contents

1 Exponential Functions

1.1 Limit

Evaluate the following limit:

$$\lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

To evaluate the limit, we first multiply the top and bottom by e^{-x} to reduce the problem. (same as dividing the top and bottom by e^x)

$$\lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} \frac{e^{-x}}{e^{-x}} = \lim_{x\to\infty} \frac{1 - e^{-2x}}{1 + e^{-2x}}$$

As $x \to \infty$, $e^{-2x} \to 0$, so we can conclude that

$$\lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} = \lim_{x\to\infty} \frac{1 - e^{-2x}}{1 + e^{-2x}} = \frac{1}{1} = 1$$

1.2 Derivative

Differentiate the function:

$$y = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Here, we simply use the quotient rule to compute the derivative:

$$y' = \frac{(e^x + e^{-x})(e^x + e^{-x}) - (e^x - e^{-x})(e^x - e^{-x})}{(e^x + e^{-x})^2}$$

At this point, students should be advised to stop, unless the problem explicitly calls for simplification.

$$y' = \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2}$$

Note: You could also expand the terms in the numerators and achieve the same result.

$$y' = \frac{(e^x + e^{-x} + (e^x - e^{-x}))(e^x + e^{-x} - (e^x - e^{-x}))}{(e^x + e^{-x})^2}$$
$$y' = \frac{(2e^x)(2e^{-x})}{(e^x + e^{-x})^2}$$
$$y' = \frac{4}{(e^x + e^{-x})^2}$$

1.3 Integral

Evaluate:

$$\int_0^1 \frac{\sqrt{1+e^{-x}}}{e^x}\,dx$$

Rewriting the integral in a more convenient fashion makes a u-substitution more evident:

$$\int_0^1 e^{-x}\sqrt{1+e^{-x}}\,dx$$
$$u = 1+e^{-x} \longrightarrow du = -e^{-x}dx$$

Note: Remember to change the limits of integration

$$\begin{aligned} &= -\int_2^{1+e^{-1}} \sqrt{u}\,du \\ &= \int_{1+e^{-1}}^2 \sqrt{u}\,du \\ &= \frac{2}{3}u^{\frac{3}{2}}\Big|_{1+e^{-1}}^2 \end{aligned}$$

and thus, the result is:

$$\int_0^1 \frac{\sqrt{1+e^{-x}}}{e^x}\,dx = \frac{2}{3}(2\sqrt{2} - (1+e^{-1})\sqrt{1+e^{-1}})$$

2 Logarithmic Functions

2.1 Limit

Evaluate the following limit:

$$\lim_{x\to\infty}[\ln(2+x) - \ln(1+x)]$$

Here, we will rely on one of the properties of the logarithmic functions, namely:

$$\lim_{x\to\infty}[\ln(2+x) - \ln(1+x)] = \lim_{x\to\infty} \ln\left(\frac{2+x}{1+x}\right)$$

Now, rewriting the argument of the natural logarithm (this trick only works when $x \neq 0$ since $x \longrightarrow \infty$):

$$\begin{aligned} &= \lim_{x\to\infty} \ln\left(\frac{(2+x)\,\frac{1}{x}}{(1+x)\,\frac{1}{x}}\right) \\ &= \lim_{x\to\infty} \ln\left(\frac{\frac{2}{x}+1}{\frac{1}{x}+1}\right) \\ &= \ln\left(\frac{0+1}{0+1}\right) = \ln(1) = 0 \end{aligned}$$

Therefore:

$$\lim_{x\to\infty}[\ln(2+x) - \ln(1+x)] = 0$$

2.2 Derivative

Find y' if:

$$x^y = y^x, \quad x, y > 0$$

Since there is no algebraic way of isolating y, we will resort to logarithmic differentiation by first applying the natural (or any) logarithm to both sides and using some of the logarithm's common properties:

$$\ln(x^y) = \ln(y^x)$$
$$y\ln(x) = x\ln(y)$$

Now, using implicit differentiation and product rule:

$$y'\ln(x) + y\frac{1}{x} = \ln(y) + x\frac{y'}{y}$$

Finally, what is left is manipulating the equation above to isolate y':

$$y'\ln(x) - x\frac{y'}{y} = \ln(y) - \frac{y}{x}$$
$$y'(\ln(x) - \frac{x}{y}) = \ln(y) - \frac{y}{x}$$
$$y' = \frac{\ln(y) - \frac{y}{x}}{\ln(x) - \frac{x}{y}}$$
$$y' = \frac{xy\ln(y) - y^2}{xy\ln(x) - x^2}$$

Note: since there is no algebraic way of explicitly solving for y, we cannot express the final solution in terms of the independent variable only, and thus, we have to leave our solution in the form shown above.

2.3 Integral

Evaluate:

$$\int \frac{x^2 + x - 1}{x + 1}\, dx$$

Notice that the degree of the polynomial in the numerator is higher than that of the polynomial in the denominator. Thus, we can use long division to simplify our integrand.

$$\begin{array}{r|l} & \quad x \\ \hline x+1 & \; x^2 + x - 1 \\ & -x^2 - x \\ \hline & \qquad\quad -1 \end{array}$$

We rewrite our integral as follows:

$$= \int \left(x + \frac{-1}{x+1}\right) dx$$
$$= \int x\, dx - \int \frac{1}{x+1}\, dx$$

Here, we see that the first integral is easy to compute, while the second integral requires a simple u-substitution (namely, $u = x + 1 \longrightarrow du = dx$):

$$\int \frac{x^2 + x - 1}{x + 1}\, dx = \frac{x^2}{2} - \ln|x + 1| + C$$

3 Inverse Trigonometric Functions

3.1 Integral

Evaluate:

$$\int \frac{x}{x^4+9}\,dx$$

This really looks like a trigonometric substitution, but we can manipulate this integral thanks to the "x" term in the numerator. To simplify the integral stated above, first notice that we can rewrite the integral and make the following substitution:

$$\int \frac{x}{x^4+9}\,dx = \int \frac{x}{(x^2)^2+9}\,dx$$
$$u = x^2 \longrightarrow du = 2x\,dx \longrightarrow \frac{du}{2} = x\,dx$$
$$= \frac{1}{2}\int \frac{du}{u^2+9}$$

Hopefully, by now you recognize this integral:

$$= \frac{1}{18}\int \frac{du}{(\frac{u}{3})^2+1}$$
$$w = \frac{u}{3} \longrightarrow dw = \frac{du}{3} \longrightarrow 3\,dw = du$$
$$= \frac{3}{18}\int \frac{dw}{w^2+1}$$
$$= \frac{1}{6}\arctan w + C$$
$$\int \frac{x}{x^4+9}\,dx = \frac{1}{6}\arctan\left(\frac{x^2}{3}\right) + C$$

4 l'Hospital's Rule

4.1 Indeterminate Limit

Find the limit:

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx}, \quad a, b \neq 0$$

It is easy to see that if we attempt to evaluate the limit, we end up with an indeterminate value (namely, 1^∞), and thus, we define the following function:

$$\text{Let} \quad y = \left(1+\frac{a}{x}\right)^{bx}$$
$$\ln(y) = \ln\left(1+\frac{a}{x}\right)^{bx}$$
$$\ln(y) = bx\ln\left(1+\frac{a}{x}\right)$$

Now we see that the limit as x approaches ∞ results in $\ln(y)$ approaching $\infty \times 0$ which is an indeterminate form that is not desirable. Thus, we decide to rearrange the equation in the following manner:

$$\ln(y) = \frac{\ln\left(1 + \frac{a}{x}\right)}{\frac{1}{bx}}$$
$$\lim_{x\to\infty} \ln(y) = \frac{0}{0}$$

Now we see that we can apply l'Hospital's Rule:

$$\lim_{x\to\infty} \ln(y) = \lim_{x\to\infty} \frac{\frac{1}{1+\frac{a}{x}} \frac{-a}{x^2}}{\frac{-1}{bx^2}} = \lim_{x\to\infty} \frac{ab}{1+\frac{a}{x}} = ab$$

Thus:

$$\lim_{x\to\infty} y = \lim_{x\to\infty} e^{\ln(y)} = e^{ab}$$
$$\lim_{x\to\infty} \left(1 + \frac{a}{x}\right)^{bx} = e^{ab}$$

5 Techniques of Integration

5.1 Integration by Parts

Evaluate:

$$\int \sin(\ln(x))\, dx$$

At this point,we have no obvious options for any sort of substitution, leaving integration by parts as the only feasible option:

$$\text{Let} \quad I = \int \sin(\ln(x))\, dx$$
$$\text{Set} \quad u = \sin(\ln(x)) \qquad dv = dx$$
$$du = \frac{\cos(\ln(x))}{x} dx \quad v = x$$
$$I = uv - \int v\, du = x\sin(\ln(x)) - \int \cos(\ln(x))\, dx$$

To evaluate the integral on the left hand side of the equation, we must now apply the integration by parts technique once more (do you see the pattern?):

$$\text{Set} \quad u = \cos(\ln(x)) \qquad dv = dx$$
$$du = \frac{-\sin(\ln(x))}{x} dx \quad v = x$$
$$I = x\sin(\ln(x)) - \left[x\cos(\ln(x)) + \int \sin(\ln(x))\, dx\right]$$
$$I = x\sin(\ln(x)) - x\cos(\ln(x)) - I$$
$$2I = x[\sin(\ln(x) - \cos(\ln(x))]$$
$$\int \sin(\ln(x))\, dx = \frac{x}{2}[\sin(\ln(x) - \cos(\ln(x))] + C$$

5.2 Trigonometric Integrals

Evaluate:

$$\int \frac{d\phi}{\cos\phi + 1}$$

To evaluate this integral, we will use an old trick, that is multiplying by the conjugate in the numerator and denominator and using the difference of squares identity (do you remember the identity?):

$$\int \frac{d\phi}{\cos\phi + 1} = \int \frac{\cos\phi - 1}{\cos^2\phi - 1}\, d\phi$$

Now, we apply the Pythagorean Identity, namely:

$$\begin{aligned}
&= \int \frac{\cos\phi - 1}{\sin^2\phi}\, d\phi \\
&= \int \frac{\cos\phi}{-\sin^2\phi}\, d\phi - \int -\csc^2\phi\, d\phi \\
&\text{Let}\quad u = \sin\phi \longrightarrow du = \cos\phi d\phi \\
&= -\int \frac{du}{u^2} - \cot\phi + C \\
&= -\int u^{-2}\, du - \cot\phi + C \\
&= u^{-1} - \cot\phi + C \\
&\int \frac{d\phi}{\cos\phi + 1} = \csc\phi - \cot\phi + C
\end{aligned}$$

5.3 Trigonometric Substitution

Evaluate:

$$\int_0^{\frac{\pi}{2}} \frac{\cos t}{\sqrt{1+\sin^2 t}}\, dt$$

At first, it is not obvious what the best approach is, however, a simplifying u-substitution helps:

$$\begin{aligned}
&\text{Let}\quad u = \sin t \longrightarrow du = \cos t\ dt \\
&= \int_0^1 \frac{du}{\sqrt{1+u^2}}
\end{aligned}$$

It is now evident that a trigonometric substitution will help us evaluate the integral:

$$\begin{aligned}
&\text{Let}\quad u = \tan\theta \longrightarrow du = \sec^2\theta\, d\theta \\
&= \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{\sqrt{1+\tan^2\theta}}\, d\theta \\
&= \int_0^{\frac{\pi}{4}} \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta|\Big|_0^{\frac{\pi}{4}} \\
&= \ln|\sqrt{2}+1| - \ln|1| \\
&\int_0^{\frac{\pi}{2}} \frac{\cos t}{\sqrt{1+\sin^2 t}}\, dt = \ln|\sqrt{2}+1|
\end{aligned}$$

Note: Here we assume that the student has the anti-derivative of the secant function committed to memory (how do you arrive to that formula?).

5.4 Partial Fractions Integral

Evaluate:

$$\int \frac{dx}{1+e^x}$$

Here, we will attempt a u-substitution since no other routes seem reasonable.

$$\text{Let} \quad u = 1 + e^x$$
$$du = e^x dx = (u-1)dx$$
$$\frac{du}{u-1} = dx$$
$$\int \frac{dx}{1+e^x} = \int \frac{du}{u(u-1)}$$

Now, to compute the right hand side, we use the partial fractions expansion:

$$\frac{1}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1}$$
$$1 = A(u-1) + Bu$$
$$For\ u = 0, \quad 1 = -A \longrightarrow A = -1$$
$$For\ u = 1, \quad B = 1$$

Thus, our integral simplifies to:

$$\int \frac{du}{u(u-1)} = -\int \frac{du}{u} + \int \frac{du}{u-1}$$
$$= -\ln(u) + \ln(u-1) + C$$
$$\int \frac{dx}{1+e^x} = -\ln|1+e^x| + \ln|e^x| + C$$

5.5 U-Substitution

Evaluate:

$$I = \int \frac{dx}{1+\sqrt[3]{x}}$$

Once again, we will use a u-substitution to simplify the problem:

$$\text{Let} \quad u = 1 + \sqrt[3]{x}$$
$$du = \frac{dx}{3(\sqrt[3]{x})^2}$$

Note here that we can use our u-substitution to back substitute for the cubic root term:

$$3(u-1)^2\, du = dx$$
$$I = \int \frac{3(u-1)^2}{u}\, du$$

As you can see, the integral becomes a simple one from here on:

$$= \int \frac{3u^2 - 6u + 3}{u}\, du$$
$$= \int (3u - 6 + \frac{3}{u})\, du$$
$$= \frac{3u^2}{2} - 6u + 3\ln|u| + C$$
$$I = \frac{3(1+\sqrt[3]{x})^2}{2} - 6(1+\sqrt[3]{x}) + 3\ln|1+\sqrt[3]{x}| + C$$

6 Improper Integrals

6.1 Infinite Bounds

Determine whether the integral converges or diverges. If it converges, evaluate the integral:

$$\int_0^\infty x^3 e^{-x^4}\, dx$$

To evaluate this integral, we must first introduce an auxiliary variable and take the bounds as limits. This is done as follows;

$$\lim_{t\to\infty}\int_0^t x^3 e^{-x^4}\, dx$$

Now, we will introduce a u-substitution to make the integral a bit more friendly:

$$\text{Let} \quad u = x^4 \longrightarrow du = 4x^3\, dx$$
$$\frac{du}{4} = x^3 dx$$

Now, we must study what happens to the limits under this substitution:

$$As \quad t \longrightarrow \infty, \quad t^4 \longrightarrow \infty$$
$$\lim_{t\to\infty}\int_0^t x^3 e^{-x^4}\, dx = \frac{1}{4}\lim_{t\to\infty}\int_0^{t^4} e^{-u}\, du$$
$$= \frac{1}{4}\lim_{t\to\infty}[-e^{-t^4} + 1]$$
$$= \frac{1}{4}[0+1] = \frac{1}{4}$$

and therefore,

$$\int_{-\infty}^\infty x^3 e^{-x^4}\, dx = \frac{1}{4}$$

6.2 Discontinuous Integrands

Determine whether the integral converges or diverges. If it converges, evaluate the integral:

$$\int_1^3 \frac{x}{(2x^2-8)^{\frac{2}{3}}}\, dx$$

Note that the integrand is discontinuous at $x = 2$. First, let us make a u-substitution to simplify the expression at hand:

$$\text{Let} \qquad u = 2x^2 - 8 \longrightarrow du = 4x\, dx \longrightarrow \frac{du}{4} = x\, dx$$
$$\int_1^3 \frac{x}{(2x^2-8)^{\frac{2}{3}}}\, dx = \frac{1}{4}\int_{-6}^{10}\frac{du}{u^{\frac{2}{3}}} = \frac{1}{4}\int_{-6}^{10} u^{-\frac{2}{3}}\, du$$

Notice that now, we must deal with the discontinuity at $u = 0$ (which comes from the original discontinuity):

$$\frac{1}{4}\int_{-6}^{10} u^{-\frac{2}{3}}\,du = \frac{1}{4}\int_{-6}^{0} u^{-\frac{2}{3}}\,du + \frac{1}{4}\int_{0}^{10} u^{-\frac{2}{3}}\,du$$
$$= \frac{1}{4}\lim_{t\to 0^-}\left[\int_{-6}^{t} u^{-\frac{2}{3}}\,du\right] + \frac{1}{4}\lim_{s\to 0^+}\left[\int_{s}^{10} u^{-\frac{2}{3}}\,du\right]$$
$$= \frac{1}{4}\lim_{t\to 0^-}\left[3u^{\frac{1}{3}}\Big|_{-6}^{t}\right] + \frac{1}{4}\lim_{s\to 0^+}\left[3u^{\frac{1}{3}}\Big|_{s}^{10}\right]$$
$$= \frac{1}{4}\lim_{t\to 0^-}\left[3(\sqrt[3]{t} - \sqrt[3]{-6})\right] + \frac{1}{4}\lim_{s\to 0^+}\left[3(\sqrt[3]{10} - \sqrt[3]{s})\right]$$

Now, we can plug in values for both t and s, and the integral converges to:

$$\int_1^3 \frac{x}{(2x^2-8)^{\frac{2}{3}}}\,dx = \frac{3}{4}[\sqrt[3]{10} + \sqrt[3]{6}]$$

6.3 Comparison Test

Determine whether this integral converges or diverges:

$$\int_1^\infty \frac{x}{x^3+1}\,dx$$

Evaluating this integral is quite difficult. However, a simple comparison will give us the answer we need since we are NOT looking to evaluate the integral:

$$\int_1^\infty \frac{x}{x^3+1}\,dx \le \int_1^\infty \frac{x}{x^3}\,dx = \int_1^\infty \frac{dx}{x^2}$$

As is (hopefully) evident, this is a convergent p-integral since $p = 2 > 1$. Thus, by comparison:

$$\int_1^\infty \frac{x}{x^3+1}\,dx \quad \text{converges}$$

7 Parametric Curves

7.1 Derivatives

Find the equation of the tangent to the curve at the given point by two methods: (a) without eliminating the parameter and (b) by first eliminating the parameter.

$$x = 1 + \sqrt{t}, \quad y = e^{t^2}\ ;\ (2, e)$$

a) First, we must find the derivative of x and y with respect to t and take the ratio of those to arrive at the needed derivative as follows:

$$\frac{dx}{dt} = \frac{1}{2\sqrt{t}}, \qquad \frac{dy}{dt} = 2te^{t^2}$$
$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2te^{t^2}}{\frac{1}{2\sqrt{t}}} = 4t^{\frac{3}{2}}e^{t^2}$$

Now, we must determine the value of the parameter t when $x = 2$:

$$x = 2 \longrightarrow 2 = 1 + \sqrt{t} \longrightarrow \sqrt{t} = 1 \longrightarrow t = 1$$

$$\left.\frac{dy}{dx}\right|_{t=1} = 4e \longrightarrow slope = 4e$$

Equation of the tangent line becomes:

$$y - e = 4e(x - 2) \longrightarrow y = e(4x - 7)$$

b) We will now attempt the same problem by eliminating the parameter. Thus, we begin by solving for t in the first parametric equation and then plugging the result into the second equation:

$$x = 1 + \sqrt{t} \longrightarrow t = (x - 1)^2 \longrightarrow y = e^{(x-1)^4}$$

$$\frac{dy}{dx} = 4(x - 1)^3 e^{(x-1)^4} \longrightarrow \left.\frac{dy}{dx}\right|_{x=2} = 4e$$

Now we see that we achieve the same equation for the tangent line to the curve (which is good!):

$$y - e = 4e(x - 2) \longrightarrow y = e(4x - 7)$$

7.2 Areas

Find the area enclosed by an ellipse using the following parametric equations:

$$x = a\cos\theta, \quad y = b\sin\theta, \quad 0 \le \theta \le 2\pi$$

To be able to calculate the area at hand, we must compute the following integral:

$$Area = \int_{x_1}^{x_2} y\,dx$$

Where y is the part of the ellipse in the first quadrant $(0 \le x \le a)$ and then multiply the result by 4 to arrive at the total area. However, note that we can use the equations we have to rewrite the integral:

$$x = a\cos\theta \longrightarrow dx = -a\sin\theta\,d\theta$$

$$\text{When} \quad x = 0,\ \theta = \frac{\pi}{2} \qquad \text{When} \quad x = a,\ \theta = 0$$

$$\int_{x_1}^{x_2} y\,dx = \int_{\frac{\pi}{2}}^{0} b\sin\theta(-a\sin\theta)\,d\theta$$

Now, we can get rid of the negative sign by flipping the bounds of integration, which will make the integral look a bit more friendly:

$$= \int_0^{\frac{\pi}{2}} ab\sin^2\theta\,d\theta = ab\int_0^{\frac{\pi}{2}} \sin^2\theta\,d\theta$$

Finally, we will use the following trigonometric identity to evaluate the integral:

$$= ab\int_0^{\frac{\pi}{2}} \frac{1}{2}(1 - cos(2\theta))\,d\theta$$

$$= ab\left[\frac{1}{2}\theta - \frac{1}{4}\sin(2\theta)\right]\Bigg|_0^{\frac{\pi}{2}} = \frac{\pi ab}{4}$$

$$\text{Area} = 4 \times \frac{\pi ab}{4} = \pi ab$$

7.3 Polar Coordinates

Find the points (in polar coordinates) on the given curve where the tangent line is horizontal or vertical

$$r = e^{\theta}$$

To be able to do so, we will first need to compute the derivative ($\frac{dy}{dx}$) using the same method we used a few problems back as well as the following equations:

$$\begin{aligned}
&x = r\cos\theta, \quad y = r\sin\theta; \\
&\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{(dr/d\theta)\sin\theta + r\cos\theta}{(dr/d\theta)\cos\theta - r\sin\theta} \\
&\frac{dr}{d\theta} = e^{\theta} \\
&\frac{dy}{dx} = \frac{e^{\theta}(\sin\theta + \cos\theta)}{e^{\theta}(\cos\theta - \sin\theta)}
\end{aligned}$$

Since e^{θ} is never equal to zero, we can simply cancel it out. From here, to find the points at which the curve has a horizontal tangent, we set the derivative equal to zero. To find vertical tangents, we will look for discontinuities in the derivative (i.e. set the denominator equal to zero):

$$\begin{aligned}
&\frac{dy}{dx} = \frac{\sin\theta + \cos\theta}{\cos\theta - \sin\theta} \\
&\text{Horizontal tangents}: \frac{dy}{dx} = 0 \longrightarrow \sin\theta + \cos\theta = 0 \\
&\sin\theta = -\cos\theta \longrightarrow \theta = \frac{3\pi}{4} + 2n\pi \ or \ \theta = \frac{7\pi}{4} + 2n\pi \\
&\text{Vertical tangents}: \cos\theta - \sin\theta = 0 \longrightarrow \cos\theta = \sin\theta \\
&\theta = \frac{\pi}{4} + 2n\pi \ or \ \theta = \frac{5\pi}{4} + 2n\pi
\end{aligned}$$

Thus, the points of interest (in the interval $0 \leq \theta \leq 2\pi$) are:

$$\begin{aligned}
&\text{Horizontal tangents}: \left(e^{\frac{3\pi}{4}}, \frac{3\pi}{4}\right) \text{ and } \left(e^{\frac{7\pi}{4}}, \frac{7\pi}{4}\right) \\
&\text{Vertical tangents}: \left(e^{\frac{\pi}{4}}, \frac{\pi}{4}\right) \text{ and } \left(e^{\frac{5\pi}{4}}, \frac{5\pi}{4}\right)
\end{aligned}$$

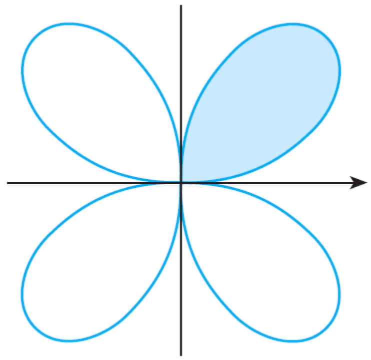
7.4 Areas in Polar Coordinates

Find the area one loop of the curve:

$$r = \sin(2\theta)$$

To do so, we must first find the values at which r goes to zero (i.e. determine the limits of integration for our problem). However, note that:

$$r = 0 \longrightarrow \sin(2\theta) = 0 \longrightarrow \theta = 0 \ and \ \theta = \frac{\pi}{2}$$



Thus, we can now compute the area using the following formula:

$$\begin{aligned}\text{Area } &= \int_a^b \frac{1}{2} r^2 \, d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{2} \sin^2(2\theta) \, d\theta \\ &= \frac{1}{4} \int_0^{\frac{\pi}{2}} (1 - \cos(4\theta)) \, d\theta \\ &= \frac{1}{4}\theta - \frac{1}{16} \sin(4\theta) \Big|_0^{\frac{\pi}{2}} \\ \text{Area in one loop } &= \frac{\pi}{8}\end{aligned}$$

8 Applications of Integration

8.1 Arc Length

Find the arc length of the curve on the given interval:

$$y = \frac{1}{3}(x^2 + 2)^{3/2}, \quad 0 \le x \le 2$$

To find the arc length, we must evaluate the following integral:

$$L = \int_0^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$

We compute the derivative to be

$$\frac{dy}{dx} = x\sqrt{x^2 + 2}$$

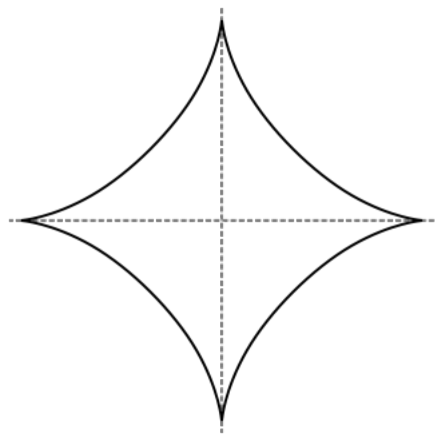
Plugging this into our formula, we get

$$\begin{aligned} L &= \int_0^2 \sqrt{1 + \left(x\sqrt{x^2+2}\right)^2} \, dx = \int_0^2 \sqrt{1 + 2x^2 + x^4} \, dx = \\ &= \int_0^2 \sqrt{(1 + x^2)^2} \, dx = \int_0^2 1 + x^2 \, dx = x + \frac{1}{3}x^3 \Big|_0^2 = 2 + \frac{8}{3} = \frac{14}{3}\end{aligned}$$

8.2 Parametric Arc Length

Find the total length of the astroid with the following parametric equations:

$$x = a\cos^3\theta, \quad y = a\sin^3\theta$$



To do so, we will use the following form of the arc length integral:

$$L = \int_a^b ds = \int_a^b \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

$$\frac{dx}{d\theta} = -3a\cos^2\theta\sin\theta, \quad \frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$$

Because of the symmetry of the problem (and one more reason, to be seen after some simplification), we will only compute the length in the first quadrant and then multiply the result by 4:

$$\begin{aligned}\frac{L}{4} &= \int_0^{\frac{\pi}{2}} \sqrt{(-3a\cos^2\theta\sin\theta)^2 + (3a\sin^2\theta\cos\theta)^2}\, d\theta \\ &= 3a\int_0^{\frac{\pi}{2}} \sqrt{\cos^4\theta\sin^2\theta + \sin^4\theta\cos^2\theta}\, d\theta \\ &= 3a\int_0^{\frac{\pi}{2}} \sqrt{(\cos^2\theta\sin^2\theta)(\cos^2\theta + \sin^2\theta)}\, d\theta \\ &= 3a\int_0^{\frac{\pi}{2}} \sqrt{\cos^2\theta\sin^2\theta}\, d\theta = 3a\int_0^{\frac{\pi}{2}} |\cos\theta\sin\theta|\, d\theta\end{aligned}$$

We pause here for a second to understand the situation. The fact that we applied the square root operation to a square quantity resulted in the appearance of the absolute value. However, since we are in the first quadrant (this is the second reason referred to earlier), the sine and cosine values are both positive, which allows us to drop the absolute values and move on by using the double angle identity from trigonometry to further simplify the integrand:

$$\begin{aligned}\frac{L}{4} &= 3a\int_0^{\frac{\pi}{2}} \cos\theta\sin\theta\, d\theta = \frac{3a}{2}\int_0^{\frac{\pi}{2}} \sin 2\theta\, d\theta \\ &= -\frac{3a}{4}\cos 2\theta\Big|_0^{\frac{\pi}{2}} = \frac{3a}{2} \\ L &= 4 \times \frac{3a}{2} = 6a\end{aligned}$$

8.3 Polar Coordinates Arc Length

Find the arc length of the following polar curve:

$$r = 2(1 + \cos\theta)$$

To compute the arc length, we will use the following form of the arc length integral:

$$L = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$
$$\frac{dr}{d\theta} = -2\sin\theta$$

To find the arc length, we will use the symmetry of the problem (and one more reason to be seen after some work) and integrate for $0 \leq \theta \leq \pi$:

$$\frac{L}{2} = \int_0^\pi \sqrt{4(1+\cos\theta)^2 + 4\sin^2\theta}\, d\theta$$
$$= \int_0^\pi \sqrt{4 + 8\cos\theta + 4\cos^2\theta + 4\sin^2\theta}\, d\theta = \int_0^\pi \sqrt{4 + 8\cos\theta + 4(\cos^2\theta + \sin^2\theta)}\, d\theta$$
$$= \int_0^\pi \sqrt{8 + 8\cos\theta}\, d\theta = 2\sqrt{2}\int_0^\pi \sqrt{1+\cos\theta}\, d\theta$$

Here, we will use the half-angle identity to simplify the integrand:

$$\frac{L}{2} = 2\sqrt{2}\int_0^\pi \sqrt{2\cos^2\left(\frac{\theta}{2}\right)}\, d\theta = 4\int_0^\pi \left|\cos\left(\frac{\theta}{2}\right)\right| d\theta$$

We pause here to understand how to deal with this absolute value that appeared. Taking the square root of a square quantity resulted in the appearance of the absolute value sign. However, notice that for $0 \leq \theta \leq \pi \longrightarrow 0 \leq \cos\left(\frac{\theta}{2}\right) \leq 1$. Since the cosine is always positive on this interval (which the reason we chose to use symmetry) we can simply drop the absolute value.

$$\frac{L}{2} = 4\int_0^\pi \cos\left(\frac{\theta}{2}\right) d\theta = 4\cdot 2\cdot \sin\left(\frac{\theta}{2}\right)\Big|_0^\pi = 8(1-0)$$
$$L = 2\times 8 = 16$$

8.4 Surface Area of Revolutions

Find the expression for the surface area of a sphere with radius r.

to be able to compute this expression, we will find the surface area of half a sphere by viewing it as the surface area of a quarter of a circle with radius r revolved about the y-axis. The equation of the circle is:

$$x^2 + y^2 = r^2$$

Now, we will use the following expression to compute the surface area:

$$SA = 2\pi\int_a^b y\, ds = 2\pi\int_a^b y\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$
$$y = \sqrt{r^2 - x^2} \longrightarrow \frac{dy}{dx} = \frac{-x}{\sqrt{r^2-x^2}}$$

Note that we only considered the positive square root when solving for y since we are restricting our circle to the first quadrant. Now that we have the expressions we need, we will evaluate the integral for $0 \leq x \leq r$ (do you see why that is?).

$$\frac{SA}{2} = 2\pi\int_0^r \sqrt{r^2-x^2}\sqrt{1+\left(\frac{-x}{\sqrt{r^2-x^2}}\right)^2}\, dx$$

Now, to simplify the integrand, we will combine the terms inside the larger square root by finding a common denominator. The results is as follows:

$$= 2\pi \int_0^r \sqrt{r^2 - x^2}\sqrt{\frac{r^2}{r^2 - x^2}}\,dx = 2\pi \int_0^r r\,dx$$

$$\frac{SA}{2} = 2\pi r \int_0^r dx = 2\pi r x\Big|_0^r = 2\pi r^2$$

$$SA = 2 \times 2\pi r^2 = 4\pi r^2$$

8.5 Surface Area of Revolution Revisited

Find the expression for the surface area of a sphere with radius r.

This time, we will attempt to compute the formula for the surface area of the sphere by rotating one quarter of the circle ($0 \le \theta \le \frac{\pi}{2}$) about the x-axis and multiplying the result by two. Also, we will do so by using the following parametric equations of the circle.

$$x = r\cos\theta, \quad y = r\sin\theta, \quad 0 \le \theta \le \frac{\pi}{2}$$

This time around, the integral that will help us arrive at the surface area is of the following form:

$$SA = 2\pi \int_a^b x\,ds = 2\pi \int_a^b x\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\,d\theta$$

$$\frac{dx}{d\theta} = -r\sin\theta, \quad \frac{dy}{d\theta} = r\cos\theta$$

$$\frac{SA}{2} = 2\pi \int_0^{\frac{\pi}{2}} r\cos\theta\sqrt{(-r\sin\theta)^2 + (r\cos\theta)^2}\,d\theta$$

$$= 2\pi \int_0^{\frac{\pi}{2}} r\cos\theta\sqrt{r^2(\sin^2\theta + \cos^2\theta)}\,d\theta$$

$$= 2\pi r^2 \int_0^{\frac{\pi}{2}} \cos\theta\,d\theta = 2\pi r^2 sin\theta\Big|_0^{\frac{\pi}{2}} = 2\pi r^2(1-0)$$

$$SA = 2 \times 2\pi r^2 = 4\pi r^2$$

9 Infinite Sequences and Series

9.1 Sequences

Determine whether the following sequences converge or diverge.

1. $a_n = \dfrac{\ln(n)}{\ln(2n)}$

   When considering any infinite sequence, we want to see the behavior as n tends towards infinity.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{\ln(n)}{\ln(2n)} = \frac{\infty}{\infty}$$

   Given the indeterminant form, we can see that this would be a good time to employ L'Hospital's Rule.

$$\lim_{n\to\infty} \frac{\ln(n)}{\ln(2n)} = \lim_{n\to\infty} \frac{\big(\ln(n)\big)'}{\big(\ln(2n)\big)'} = \lim_{n\to\infty} \frac{1/n}{2/2n} = \lim_{n\to\infty} \frac{n}{n} = 1$$

This tells us that our sequence is convergent.

2. $b_n = \dfrac{\sin(2n)}{1+\sqrt{n}}$
By imagining what happens to $\sin(2n)$ as n grows, we can see that $\sin(2n)$ oscillates between -1 and 1. Knowing this, let us consider the following.

$$\frac{-1}{1+\sqrt{n}} \leq \frac{\sin(2n)}{1+\sqrt{n}} \leq \frac{1}{1+\sqrt{n}}$$

Again, we are concerned with the behavior of our sequences as n approaches infinity.

$$\lim_{n\to\infty} \frac{-1}{1+\sqrt{n}} \leq \lim_{n\to\infty} \frac{\sin(2n)}{1+\sqrt{n}} \leq \lim_{n\to\infty} \frac{1}{1+\sqrt{n}}$$

$$0 \leq \lim_{n\to\infty} \frac{\sin(2n)}{1+\sqrt{n}} \leq 0$$

Thus, our sequence converges to zero by squeeze theorem.

9.2 Series

Determine whether the following series is convergent or divergent. If it is convergent, find its sum.

$$\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$$

When testing to see if series are divergent, one of the easiest tests to use is the Test for Divergence. We can clearly see that the argument of the sum approaches 0 as n approaches infinity. What does this mean? Inconclusive. Do not make the mistake of trying to use the converse of the Test for Divergence. The Harmonic Series is a perfect example of this not working.
What we can see is the following:

$$\frac{3}{n(n+3)} = \frac{3}{n^2+3n} \leq \frac{3}{n^2}$$

Now the right hand side is convergent by p-test. Since each of these terms and our original terms are all positive, our series is then convergent by comparison test.

To find the sum of this series we'll have to work a bit harder. The series is not a geometric series so we don't have a convenient formula for the sum. Instead, we will examine the partial sums. We can find its partial fraction composition to see if that helps before we tackle the partial sums.

$$\frac{3}{n(n+3)} = \frac{A}{n} + \frac{B}{n+3}$$

$$3 = A(n+3) + B(n)$$

We can see by two easy substitutions that $A = 1$ and $B = -1$. So

$$\sum_{n=1}^{\infty} \frac{3}{n(n+3)} = \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+3}$$

$$s_n = \left(\frac{1}{1} - \frac{1}{4}\right) + \left(\frac{1}{2} - \frac{1}{5}\right) + \left(\frac{1}{3} - \frac{1}{6}\right) + \left(\frac{1}{4} - \frac{1}{7}\right) + \left(\frac{1}{5} - \frac{1}{8}\right) + ... + \left(\frac{1}{n} - \frac{1}{n+3}\right)$$

Note that terms begin to cancel. And, in fact, our partial sum collapses.

$$s_n = \left(\frac{1}{1} - \cancel{\frac{1}{4}}\right) + \left(\frac{1}{2} - \cancel{\frac{1}{5}}\right) + \left(\frac{1}{3} - \cancel{\frac{1}{6}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{7}}\right) + \left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{8}}\right) + ... + \left(\cancel{\frac{1}{n}} - \frac{1}{n+3}\right)$$

$$s_n = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} - \frac{1}{n+3}$$

Finally, we know that if the limit of the partial sum exists, then it is the sum of our series.

$$\sum_{n=1}^{\infty} \frac{3}{n(n+3)} = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1}{1} + \frac{1}{2} + \frac{1}{3} - \frac{1}{n+3} = 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}$$

9.3 Integral Test

Determine whether the series is convergent or divergent.

$$\sum_{n=2}^{\infty} \frac{1}{n \ln^2 n}$$

The first detail we can notice is that the argument goes to 0 as n approaches infinity. Next, we consider what tests to use. This appears to be a reasonable Integral Test application. It is important to note that the argument is positive and decreasing for all n. So we now take the integral.

$$\int_2^{\infty} \frac{dx}{x \ln^2 x} = \lim_{t\to\infty} \int_2^t \frac{dx}{x \ln^2 x}$$

Let $u = \ln(x)$. So $du = \frac{1}{x}dx$

$$\lim_{t\to\infty} \int_2^t \frac{dx}{x \ln^2 x} = \lim_{t\to\infty} \int_{\ln(2)}^{\ln(t)} \frac{du}{u^2} = \lim_{t\to\infty} \left.\frac{-1}{x}\right|_{\ln(2)}^{\ln(t)} = \lim_{t\to\infty} \frac{-1}{\ln(t)} + \frac{1}{\ln(2)} = \frac{-1}{\infty} + \frac{1}{\ln(2)} < \infty$$

Now, by the Integral Test,

$$\sum_{n=2}^{\infty} \frac{1}{n \ln^2 n} \text{ converges } \iff \int_2^{\infty} \frac{dx}{x \ln^2 x} \text{ converges.}$$

Therefore, our summation converges.

9.4 Limit Comparison Test

Determine whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{1}{n^{(1+1/n)}}$$

When we see summation that incorporate strange terms such as this, we drastically limit which tests we can employ. In this case, we will attempt to solve this with the Limit Comparison Test.

$$\sum_{n=1}^{\infty} \frac{1}{n^{(1+1/n)}} = \sum_{n=1}^{\infty} \frac{1}{n \sqrt[n]{n}}$$

We have two pieces to work with here. If we were to choose $b_n = \frac{1}{\sqrt[n]{n}}$ then the resulting quotient will be the harmonic sequence which goes to 0. Knowing that a limit of 0 is inconclusive, we'll try the other piece. Let $b_n = \frac{1}{n}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n\sqrt[n]{n}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{\sqrt[n]{n}}$$

$$\text{Let } y = \frac{1}{\sqrt[n]{n}} \text{ so } \lim_{n\to\infty} y = \lim_{n\to\infty} \tfrac{1}{\sqrt[n]{n}} = \lim_{n\to\infty} n^{-1/n}$$

$$\lim_{n\to\infty} \ln(y) = \lim_{n\to\infty} \frac{-1}{n}\ln(n) = \lim_{n\to\infty} -\frac{\ln(n)}{n} \overset{[\frac{\infty}{\infty}]}{\underset{\text{L'H}}{=}} \lim_{n\to\infty} -\frac{1}{n} = 0$$

$$\text{So } \lim_{n\to\infty} \ln(y) = 0 \text{ then } \lim_{n\to\infty} y = \lim_{n\to\infty} \frac{a_n}{b_n} = e^0 = 1 > 0$$

By Limit Comparison Test, both series diverge or both series converge. b_n was the harmonic sequence and therefore, its series would diverge. Thus, our original series diverges.

9.5 Alternating Series

Determine whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} (-1)^n(\sqrt{n+1} - \sqrt{n})$$

This alternating series poses a few problems that prevent us from trying most of our tests, but fortunately, we simply use the Alternating Series Test. First, we must see if the sequence without the alternator, $b_n = (\sqrt{n+1} - \sqrt{n})$ is decreasing. Since this is not trivial, let us use some calculus for $f(x) = (\sqrt{x+1} - \sqrt{x})$.

$$f'(x) = \frac{1}{2\sqrt{x+1}} - \frac{1}{2\sqrt{x}} = \frac{\sqrt{x} - \sqrt{x+1}}{2\sqrt{x}\sqrt{x+1}}$$

$f(x)$ then has critical values at $x = 0$ and $x = -1$. Since $x = -1$ is outside our domain, we can ignore it. Similarly, we are only concerned with $n > 1$ so can consider our function monotone for all $n > 0$. By plugging in any positive x value, it is easy to see that the denominator is always positive while the numerator is negative. Thus $f(x)$ is decreasing and our sequence will decrease for all n. Now we must examine the limit.

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n}) = \infty - \infty$$

This is an indeterminate form but we've got plenty of practice with these limits. Let us use the conjugate method.

$$\lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n}) = \lim_{n\to\infty} \frac{(\sqrt{n+1} - \sqrt{n})}{1} \cdot \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}} = \lim_{n\to\infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} = 0$$

Then, by the Alternating Series Test, our series is convergent.

9.6 Root Test

Determine whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \left(\frac{-2n}{n+1}\right)^{5n}$$

The exponent is a red flag that we should use the Root Test.

$$\lim_{n\to\infty} \sqrt[n]{\left|\left(\frac{-2n}{n+1}\right)^{5n}\right|} = \lim_{n\to\infty} \sqrt[n]{\left(\left|\frac{-2n}{n+1}\right|\right)^{5n}} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{2n}{n+1}\right)^{5n}}$$

$$= \lim_{n\to\infty} \left(\frac{2n}{n+1}\right)^5 = \lim_{n\to\infty} \frac{2^5 n^5}{(n+1)^5} = 2^5 = 32 > 1$$

Then, by the Root Test, the summation is divergent.

9.7 Ratio Test

Determine whether the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{(-10)^n}{4^{2n+1}(n+1)}$$

With several components to juggle, Ratio Test is always a strong option.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{(-10)^{n+1}}{4^{2(n+1)+1}(n+1+1)}}{\frac{(-10)^n}{4^{2n+1}(n+1)}}\right| = \lim_{n\to\infty} \left|\frac{(-10)^{n+1}}{4^{2n+3}(n+2)} \cdot \frac{4^{2n+1}(n+1)}{(-10)^n}\right| =$$

$$= \lim_{n\to\infty} \frac{(10)^{n+1}}{4^{2n+3}(n+2)} \cdot \frac{4^{2n+1}(n+1)}{(10)^n} = \lim_{n\to\infty} \frac{10^n 10}{4^{2n+1}4^2(n+2)} \cdot \frac{4^{2n+1}(n+1)}{10^n}$$

$$= \lim_{n\to\infty} \frac{10(n+1)}{4^2(n+2)} = \frac{10}{16} = \frac{5}{8} < 1$$

Then, by Ratio Test, our series is absolutely convergent and therefore, convergent.

9.8 Power Series

Find the radius of convergence and the interval of convergence of the series:

$$\sum_{n=1}^{\infty} \frac{x^n}{1 \cdot 3 \cdot 5 \cdot ... \cdot (2n-1)}$$

To determine the radius of convergence, we simply use the ratio test to arrive at an inequality involving x:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{1 \cdot 3 \cdot 5 \cdot ... \cdot (2(n+1)-1)} \cdot \frac{1 \cdot 3 \cdot 5 \cdot ... \cdot (2n-1)}{x^n}\right|$$

We now see that many terms cancel out for the quotient to now become:

$$= \lim_{n\to\infty} \left|\frac{\cancel{x^{n+1}}}{\cancel{1} \cdot \cancel{3} \cdot \cancel{5} \cdot ... \cdot \cancel{(2n-1)} \cdot (2(n+1)-1)} \cdot \frac{\cancel{1} \cdot \cancel{3} \cdot \cancel{5} \cdot ... \cdot \cancel{(2n-1)}}{\cancel{x^n}}\right| = \lim_{n\to\infty} \left|\frac{x}{2(n+1)-1}\right|$$

Now, since we know that $n \in [1, \infty)$, we can drop the absolute values from that term. Also, since the limit is independent of x, we can pull the $|x|$ outside the limit

$$= \lim_{n\to\infty} \left|\frac{x}{2(n+1)-1}\right| = |x| \lim_{n\to\infty} \frac{1}{2n+1} = |x| \cdot 0 = 0 \text{ for all } x \text{ values.}$$

Thus, since the series converges no matter what the x value is:

$$R = \infty, \text{ Interval of convergence} : (-\infty, \infty)$$

9.9 More Power Series

Find the radius of convergence and the interval of convergence of the series:

$$\sum_{n=1}^{\infty} \frac{n}{b^n}(x-a)^n, \quad b > 0$$

As is always the case, we will use the ratio test to determine the radius of convergence as well as the interval of convergence:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)(x-a)^{n+1}}{b^{n+1}}\frac{b^n}{n(x-a)^n}\right|$$
$$= \lim_{n\to\infty}\left|\frac{(n+1)(x-a)^{\cancel{n+1}}}{b^{\cancel{n+1}}}\frac{\cancel{b^n}}{n\cancel{(x-a)^n}}\right| = \lim_{n\to\infty}\left|\frac{n+1}{bn}(x-a)\right|$$

Now, since the limit does not affect x nor a, we can pul the $(x-a)$ term outside the limit (do not forget about the absolute value!) and the limit becomes:

$$= |x-a| \lim_{n\to\infty}\left|\frac{n+1}{bn}\right| = |x-a|\left|\frac{1}{b}\right|$$

Since $b > 0$, we can simply remove the absolute value signs from around $\frac{1}{b}$. Also, since the convergence of the series at hand requires the ratio test to result in a value less than 1, we will impose that restriction and see what happens:

$$\frac{1}{b}|x-a| < 1 \longrightarrow |x-a| < b \longrightarrow \text{ Radius of convergence} = b$$

From the radius of convergence, we see that the series will converge for all values of x in the interval $a-b < x < a+b$. However, we need to check whether or not the series converges at the boundaries of the interval shown. Thus, we plug in the corresponding values and test for convergence/divergence:

For $x = a - b$:

$$\sum_{n=1}^{\infty} \frac{n}{b^n}((a-b)-a)^n = \sum_{n=1}^{\infty} \frac{n}{b^n}(-b)^n = \sum_{n=1}^{\infty}(-1)^n n \text{ Diverges by the Test for Divergence}$$

For $x = a + b$:

$$\sum_{n=1}^{\infty} \frac{n}{b^n}((a+b)-a)^n = \sum_{n=1}^{\infty} \frac{n}{b^n}(b)^n = \sum_{n=1}^{\infty} n \text{ Diverges by the Test for Divergence}$$

Thus Interval of Convergence $= (a-b, a+b)$

9.10 Maclaurin Series

Find the Maclaurin series for the following function:

$$f(x) = \sin^2 x$$

To complete this problem, one is tempted to use the Maclaurin series for the sine function and then squaring it. However, expanding an infinite series after having squared it is very difficult. However, we can utilize the following identity for another approach:

$$\sin^2 x = \frac{1}{2}(1 - \cos(2x))$$

Knowing that the Maclaurin expansion of the cosine function is the following will also be important:

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \longrightarrow \cos 2x = \sum_{n=0}^{\infty} (-1)^n \frac{(2x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{2^{2n} x^{2n}}{(2n)!}$$

$$-\frac{1}{2}\cos 2x = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{2^{2n} x^{2n}}{2(2n)!} = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{2^{2n-1} x^{2n}}{(2n)!}$$

Note that when $n = 0$, the values of the last summation above is $-\frac{1}{2}$. Thus, $\frac{1}{2}$ is added, the index will simply start from 1 rather than 0.

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^{2n-1} x^{2n}}{(2n)!}$$