Contents

1 Limits

1.1 Basic Factoring Example

$$\lim_{x\to 1} \frac{x^3 - 1}{x^2 - 1}$$

Recognize the denominator as a difference of squares, since you want to cancel the $(x - 1)$ factor on the bottom you should factor out an $(x - 1)$ from the top either realizing it as a difference of cubes or simply doing long division:

$$= \lim_{x\to 1} \frac{(x - 1)(x^2 + x + 1)}{(x - 1)(x + 1)}$$

Cancel a factor of $(x - 1)$ on top and bottom:

$$\begin{aligned} &= \lim_{x\to 1} \frac{x^2 + x + 1}{x + 1} \\ &= \frac{1 + 1 + 1}{1 + 1} \\ &\boxed{= \frac{3}{2}} \end{aligned}$$

1.2 One-Sided Limit

$$\lim_{x\to 0} \frac{\sqrt{x^2}}{x}$$

Recognize the numerator as the $|x|$ and rewrite:

$$\lim_{x\to 0} \frac{|x|}{x}$$

Have to rewrite the absolute value as a piecewise function:

$$|x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$

Now consider the two one-sided limits:

$$\lim_{x\to 0^-} \frac{-x}{x} \text{ and } \lim_{x\to 0^+} \frac{x}{x}$$

It is now clear that the left-handed limit goes to -1 and the right-handed limit goes to 1, thus the limit DNE

$$\boxed{\lim_{x\to 0} \frac{\sqrt{x^2}}{x} \text{ DNE}}$$

1.3 Squeeze Theorem

$$\lim_{x\to 0} x\sin\left(\frac{1}{x}\right)$$

Recognize the limit as a 0 times a function that is bounded which indicates that the Squeeze Thm may be useful. Setup the inequality that represents the fact that the sine function is bounded:

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

Now manipulate the inequality until you arrive at the original problem; multiply everything by an x:

$$-x \leq x\ \sin\left(\frac{1}{x}\right) \leq x$$

Now take the limit of all three parts:

$$\lim_{x\to 0} -x \leq \lim_{x\to 0} x\ \sin\left(\frac{1}{x}\right) \leq \lim_{x\to 0} x$$

$$0 \leq \lim_{x\to 0} x\ \sin\left(\frac{1}{x}\right) \leq 0$$

$$\implies \boxed{\lim_{x\to 0} x\ \sin\left(\frac{1}{x}\right) = 0}$$

1.4 Rationalizing

$$\lim_{x\to\infty} -x + \sqrt{x^2+ax} \text{ , where a is a positive constant}$$

$\infty - \infty$ case, so try rationalizing:

$$\lim_{x\to\infty} -x + \sqrt{x^2+ax} \cdot \frac{-x-\sqrt{x^2+ax}}{-x-\sqrt{x^2+ax}}$$
$$\lim_{x\to\infty} \frac{x^2-(x^2+ax)}{-x-\sqrt{x^2+ax}}$$
$$\lim_{x\to\infty} \frac{-ax}{-x-\sqrt{x^2+ax}}$$

Now multiply top and bottom by the reciprocal of the highest power of x in the denominator; here multiply by one over x:

$$\lim_{x\to\infty} \frac{-ax}{-x-\sqrt{x^2+ax}} \cdot \frac{\frac{1}{x}}{\frac{1}{x}}$$
$$\lim_{x\to\infty} \frac{-a}{-1-\sqrt{1+\frac{a}{x}}}$$
$$\boxed{=\frac{a}{2}}$$

1.5 Limits using Trig. Identities

$$\lim_{u\to 0} \frac{\sin(a+u) - \sin(a)}{u}$$

Expand using the addition formula for sine:

$$\lim_{u\to 0} \frac{\sin(a)\cos(u) + \cos(a)\sin(u) - \sin(a)}{u}$$
$$= \lim_{u\to 0} \frac{\sin(a)(\cos(u) - 1) + \sin(u)\cos(a)}{u}$$

The key is to use the following identities:

$$\lim_{u\to 0} \frac{\sin(u)}{u} = 1 \quad \text{and} \quad \lim_{u\to 0} \frac{1-\cos(u)}{u} = 0$$

Then,

$$= \lim_{u\to 0} \left(-\sin(a)\frac{1-\cos(u)}{u} + \cos(a)\frac{\sin(u)}{u} \right)$$
$$\boxed{= \cos(a)}$$

1.6 Limits involving Infinity, Part I

$$\lim_{n\to\infty} \frac{1+2+3+\cdots+n}{n^2}$$

Recall that:

$$1+2+3+\cdots+n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\lim_{n\to\infty} \frac{n(n+1)}{2n^2}$$

Divide top and bottom by the highest power of n in the denominator:

$$\lim_{n\to\infty} \frac{n^2+n}{2n^2} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{1+\frac{1}{n}}{2}$$

$$\boxed{= \frac{1}{2}}$$

Recall the three limit cases for rational functions where $x \longrightarrow \infty$:

When the degree of the numerator equals the degree of the denominator then you can "read off" the answer, in practice this amounts to dividing top and bottom by the highest power of x in the denominator, e.g.

$$\lim_{x\to\infty} \frac{8x^3+5x+1}{17x^3-2} = \frac{8}{17}$$

When the degree of the numerator is greater than the degree of the denominator then the answer will be infinite, e.g.

$$\lim_{x\to\infty} \frac{15x^2-15}{5x+729} = \infty$$

And when the degree of the denominator exceeds that of the numerator then the limit will go to zero, e.g.

$$\lim_{x\to\infty} \frac{5x+729}{15x^2-15} = 0$$

1.7 Limits involving Infinity, Part II

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

In the theory of relativity, the mass of a particle with velocity v is given by the equation above. Find the mass as $v \longrightarrow c^-$.

$$\lim_{v \to c^-} \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\text{As } v \longrightarrow c^-, \frac{v^2}{c^2} \longrightarrow 1$$

$$\text{Thus, } \sqrt{1 - \frac{v^2}{c^2}} \longrightarrow 0$$

$$\text{And,} \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \longrightarrow \infty$$

$$\boxed{\lim_{v \to c^-} \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} = \infty}$$

1.8 Continuity

A function f is defined as follows:

$$f(x) = \begin{cases} \sin(x) & \text{if } x \leq \pi \\ ax + b & \text{if } \pi < x \leq 5 \\ x^2 + b & x > 5 \end{cases}$$

where *a,b* are constants. Determine *a,b* such that the function *f(x)* is continuous everywhere.

Geometrically a piecewise function is continuous if at the point where the functions switch (i.e. at π and 10, in our case) the two one-sided limits equal each other and the limit value equals the function values at that point. Because both the sine function and polynomials are continuous everywhere, we need only make sure the two one-sided limits are equal (then the function values will automatically be the same). So we need:

$$\lim_{x\to\pi^-} \sin(x) = \lim_{x\to\pi^+} (ax+b) \text{ and } \lim_{x\to 5^-} (ax+b) = \lim_{x\to 5^+} (x^2+b)$$

From the first equation:

$$\sin(\pi) = 0 = a\pi + b$$
$$a = -\frac{b}{\pi}$$

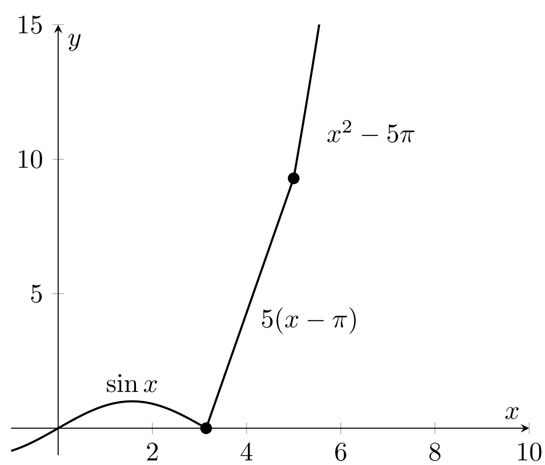
From the second equation:

$$a(5) + b = (5)^2 + b$$

Plug in the expression for a in terms of b

$$5\frac{-b}{\pi} + b = 25 + b$$

$$\boxed{b = -5\pi} \quad \text{and} \quad \boxed{a = -\frac{-5\pi}{\pi} = 5}$$



1.9 Precise definition of a Limit - Linear Case

Prove that $\lim_{x \to c}(ax+b) = ac+b$

Given $\epsilon > 0$ find $\delta > 0$ s.t.

if $0 < |x-c| < \delta$ then $|(ax+b) - ac - b| < \epsilon$

Scratch Work:

$$|ax+b-ac-b| < \epsilon$$
$$= |ax - ac| < \epsilon$$
$$= |a(x-c)| < \epsilon$$
$$= |a||x-c| < \epsilon$$
$$= |x-c| < \frac{\epsilon}{|a|}$$
$$\implies \text{choose } \delta = \frac{\epsilon}{|a|}$$

Formal:

$$0 < |x-c| < \delta$$
$$0 < |a||x-c| < |a|\delta$$
$$0 < |a(x-c)| < |a|\delta$$
$$0 < |ax - ac + b - b| < |a|\left(\frac{\epsilon}{|a|}\right)$$
$$|(ax+b) - ac - b| < \epsilon \quad \square$$

Note that if $a = 0$, then you have the constant function $y = b$, in this case an $\epsilon - \delta$ proof is unnecessary:

$$\lim_{x \to c} b = b$$

Because b is a constant it is unaffected by the limit and can be pulled outside:

$$b \cdot \lim_{x \to c} 1 = b$$
$$\therefore b = b$$

2 Derivatives

2.1 Limit Definition, Part I (i.e. x+h definition)

Using the limit definition, find the derivative of $f(x) = \dfrac{1}{1+\sqrt{x}}$

$$\lim_{h\to 0} \frac{\frac{1}{1+\sqrt{x+h}} - \frac{1}{1+\sqrt{x}}}{h}$$

It is easiest to first create a common denominator on the top and then rationalize the expression:

$$\lim_{h\to 0} \frac{1+\sqrt{x}-(1+\sqrt{x+h})}{h(1+\sqrt{x+h})(1+\sqrt{x})}$$
$$\lim_{h\to 0} \frac{\sqrt{x}-\sqrt{x+h}}{h(1+\sqrt{x+h})(1+\sqrt{x})} \cdot \frac{\sqrt{x}+\sqrt{x+h}}{\sqrt{x}+\sqrt{x+h}}$$
$$\lim_{h\to 0} \frac{x-(x+h)}{h(1+\sqrt{x+h})(1+\sqrt{x})(\sqrt{x}+\sqrt{x+h})}$$

(Note that it is almost always easiest to leave the denominator unexpanded)

$$\lim_{h\to 0} \frac{-h}{h(1+\sqrt{x+h})(1+\sqrt{x})(\sqrt{x}+\sqrt{x+h})}$$
$$\lim_{h\to 0} \frac{-1}{(1+\sqrt{x+h})(1+\sqrt{x})(\sqrt{x}+\sqrt{x+h})}$$

Now you can plug $h = 0$ in:

$$\boxed{= \frac{-1}{(1+\sqrt{x})^2(2\sqrt{x})}}$$

2.2 Limit Definition, Part II (i.e. x-a defintion)

Using the limit definition, find the derivative of $f(x) = \dfrac{x}{x-1}$

$$\lim_{x\to a} \frac{\frac{x}{x-1} - \frac{a}{a-1}}{x-a}$$

Make a common denominator on the top and work towards canceling the problem factor of $x - a$ on the bottom

$$\lim_{x\to a} \frac{x(a-1) - a(x-1)}{(x-1)(a-1)(x-a)}$$
$$\lim_{x\to a} \frac{xa - x - ax + a}{(x-1)(a-1)(x-1)}$$
$$\lim_{x\to a} \frac{a-x}{(x-1)(a-1)(x-a)}$$

Pull a negative one out of the $a - x$ factor on the bottom

$$\lim_{x\to a} \frac{-1}{(x-1)(a-1)}$$
$$\boxed{= \frac{-1}{(a-1)^2}}$$

2.3 Chain and Product Rule

Calculate the derivative of $f(x) = \sin(\cos^2 x) \cdot \cos(\sin^2 x)$

Using the product rule we get:

$$\begin{aligned} f'(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\left[\sin(\cos^2(x))\right]\cos(\sin^2(x)) + \sin(\cos^2(x))\frac{\mathrm{d}}{\mathrm{d}x}\left[\cos(\sin^2(x))\right] \\ &= \cos(\cos^2(x)) \cdot \big[-2\sin(x)\cos(x)\cos(\sin^2(x))\big] + \sin(\cos^2(x))\sin(\sin^2(x)) \cdot \big[-2\sin(x)\cos(x)\big] \\ &= -2\sin(x)\cos(x)\Big(\cos(\cos^2(x))\cos(\sin^2(x)) + \sin(\cos^2(x))\sin(\sin^2(x))\Big) \end{aligned}$$

If you want to simplify use the double angle formula for sine and the trig. addition formula for cosine

$$\boxed{= -\sin(2x)\cos(\cos^2 x - \sin^2 x)}$$

2.4 Quotient Rule

Calculate the derivative of $f(x) = \dfrac{\sin^2 x}{\sin(x^2)}$

$$\begin{aligned} f'(x) &= \frac{\sin(x^2)(2\sin(x)\cos(x)) - \sin^2(x)\cos(x^2)(2x)}{\sin^2(x^2)} \\ &= 2\sin(x)\left(\frac{\sin(x^2)\cos(x) - s\sin(x)\cos(x^2)}{\sin(x^2)}\right) \\ &= 2\sin(x)\left(\frac{\cos(x)}{\sin(x^2)} - \frac{x\sin(x)\cos(x^2)}{\sin^2(x^2)}\right) \\ &\boxed{= 2\sin(x)\csc(x^2)\Big(\cos(x) - x\sin(x)\cot(x^2)\Big)} \end{aligned}$$

2.5 Ex. involving all types of derivative techniques

Calculate the derivative of $f(x) = \dfrac{x\sin^2(2x)}{(1+x^2)^2}$

$$\begin{aligned}
f'(x) &= \frac{(1+x^2)^2\frac{\mathrm{d}}{\mathrm{d}x}\left[x\sin^2(2x)\right] - x\sin^2(2x)\frac{\mathrm{d}}{\mathrm{d}x}\left[(1+x^2)^2\right]}{\left[(1+x^2)^2\right]^2} \\
&= \frac{(1+x^2)^2\cdot\big(\sin^2(2x) + 2x\sin(2x)\cos(2x)(2)\big) - x\sin^2(2x)\cdot 2(1+x^2)(2x)}{(1+x^2)^4} \\
&= \frac{(1+x^2)^2\cdot\big(\sin^2(2x) + 4x\sin(2x)\cos(2x)\big) - x\sin^2(2x)\cdot 4x(1+x^2)}{(1+x^2)^4} \\
&= \frac{(1+x^2)^2\cdot\big(\sin^2(2x) + 2x\sin(4x)\big) - 4x^2\sin^2(2x)(1+x^2)}{(1+x^2)^4}
\end{aligned}$$

To simplify, factor out a $(1+x^2)$ term from the numerator

$$= \boxed{\frac{(1+x^2)\cdot\big(\sin^2(2x) + 2x\sin(4x)\big) - 4x^2\sin^2(2x)}{(1+x^2)^3}}$$

2.6 Derivatives w/ Fractional Exponents

Differentiate $\dfrac{x^{1/5}}{1+x^{-4/5}}$

$$
\begin{aligned}
&\frac{(1+x^{-4/5})\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{1/5}\right)-x^{1/5}\cdot\frac{\mathrm{d}}{\mathrm{d}x}\left(1+x^{-4/5}\right)}{(1+x^{-4/5})^2}\\
&=\frac{(1+x^{-4/5})(\frac{1}{5}x^{-4/5})-x^{1/5}(-\frac{4}{5}x^{-9/5})}{(1+x^{-4/5})^2}\\
&=\frac{x^{-4/5}+x^{-8/5}+4x^{-8/5}}{5(1+x^{-4/5})^2}\\
&=\frac{x^{-4/5}+5x^{-8/5}}{5(1+x^{-4/5})^2}\\
&=\frac{1+5x^{-4/5}}{5x^{4/5}(1+x^{-4/5})^2}\cdot\frac{x^{4/5}}{x^{4/5}}\\
&=\frac{x^{4/5}+5}{5x^{8/5}(1+x^{-4/5})^2}\\
&\boxed{=\frac{x^{4/5}+5}{5(x^{4/5}+1)^2}}
\end{aligned}
$$

3 Integrals

3.1 Riemann sums

Express the integral as a limit of Riemann sums, then evaluate.

$$\int_0^2 (x^2 + x + 1)\, dx$$
$$\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}$$
$$x_i = a + i\Delta x = 0 + i(\frac{2}{n}) = \frac{2i}{n}$$

Note that if using right-hand sums then i=1,2,...,n. If using left-hand sums then i=0,1,...,n-1. And if using midpoints then i=$\frac{1}{2}, \frac{3}{2}, ..., \frac{2n-1}{2}$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i^2 2^2}{n^2} + \frac{2i}{n} + 1 \right) \frac{2}{n}$$
$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{i^2 2^3}{n^3} + \frac{i2^2}{n^2} + \frac{2}{n} \right)$$
$$= \lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{i^2 2^3}{n^3} + \sum_{i=1}^{n} \frac{i2^2}{n^2} + \sum_{i=1}^{n} \frac{2}{n} \right)$$
$$= \lim_{n\to\infty} \left(\frac{2^3}{n^3} \sum_{i=1}^{n} i^2 + \frac{2^2}{n^2} \sum_{i=1}^{n} i^1 + \frac{2}{n} \sum_{i=1}^{n} i^0 \right)$$

use the appropriate formulae for the summations:

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \; and \; \sum_{i=1}^{n} i^1 = \frac{n(n+1)}{2} \; and \; \sum_{i=1}^{n} i^0 = n$$

then,

$$= \lim_{n\to\infty} \left(\frac{2^3 n(n+1)(2n+1)}{6n^3} + \frac{2^2 n(n+1)}{2n^2} + \frac{2n}{n} \right)$$

to simplify calculations simply notice that the degree of the numerators of the fractions are the same as the denominators, thus just look at the coefficients of the leading terms:

$$\boxed{= \frac{2^3}{3} + \frac{2^2}{2} + 2}$$

3.2 U-substitution, basic

$$\int_{-2}^{-4} (x+4)^{10}\,\mathrm{d}x$$

$$u = x + 4$$

$$\mathrm{d}u = \mathrm{d}x$$

Changing the limits of integration yields:

$$u : 2 \longrightarrow 0$$

$$\int_{2}^{0} u^{10}\,\mathrm{d}u$$

$$= \frac{u^{11}}{11}\bigg|_{2}^{0}$$

$$= \boxed{\frac{-2^{11}}{11}}$$

3.3 Indefinite integral, u-substitution, basic manipulation of substitution

$$\int x^3\sqrt{x^2+1}\,\mathrm{d}x$$

In general it is good to make u equal to whatever is beneath the square root. Then notice that you will want to separate the x^3 into $x^2 \cdot x$

$$u = x^2 + 1 \longrightarrow u - 1 = x^2$$
$$\mathrm{d}u = 2x\mathrm{d}x$$

Plugging the substitution into the integral yields

$$\frac{1}{2}\int (u-1)\sqrt{u}\,\mathrm{d}u$$
$$= \frac{1}{2}\int (u^{\frac{3}{2}} - u^{\frac{1}{2}})\,\mathrm{d}u$$
$$= \frac{1}{2}\Big(\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}}\Big) + C$$

After evaluating an indefinite integral you need to always substitute back to the original variable; in this case the answer needs to be expressed in terms of x

$$\boxed{= \frac{(x^2+1)^{\frac{5}{2}}}{5} - \frac{(x^2+1)^{\frac{3}{2}}}{3} + C}$$

3.4 Definite integral, u-substitution, basic manipulation of substitution

$$\int_0^4 \frac{x}{\sqrt{1+2x}}\,\mathrm{d}x$$

Since the integral can't be evaluated directly, make the substitution equal to the radicand in hopes of simplifying the integrand

$$\begin{aligned} &u = 1+2x \;\longrightarrow\; x = \frac{u-1}{2} \\ &\mathrm{d}u = 2\mathrm{d}x \end{aligned}$$

Also change the limits of integration for x to limits of integration for u by plugging the lower and upper x-limits into the substitution:

$$\text{Lower Limit}: \quad u = 1+2(0) = 1 \quad , \quad \text{Upper Limit}: \quad u = 1+2(4) = 9$$

$$\begin{aligned} &\int_1^9 \frac{u-1}{2}\Big(\frac{1}{\sqrt{u}}\Big)\frac{\mathrm{d}u}{2} \\ &= \frac{1}{4}\int_1^9 \frac{u-1}{\sqrt{u}}\,\mathrm{d}u \\ &= \frac{1}{4}\int_1^9 \left(u^{\frac{1}{2}} - u^{-\frac{1}{2}}\right)\mathrm{d}u \\ &= \frac{1}{4}\left(\frac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right)\Big|_1^9 \\ &= \frac{1}{2}\left(\frac{27}{3} - 3 - (\frac{1}{3} - 1)\right) \\ &= \frac{1}{2}\left(\frac{18}{3} + \frac{2}{3}\right) \\ &\boxed{= \frac{10}{3}} \end{aligned}$$

3.5 Integrals w/ Symmetric Limits

$$I = \int_{-1}^{1} \frac{\sin(x)}{1+x^2}\,\mathrm{d}x$$

This integral cannot be evaluated using only Calc I techniques, instead notice that you are integrating an odd function over symmetric limits $\implies$ I = 0.

First check that the integrand is indeed odd (i.e. $f(-x) = -f(x)$)

$$f(x) = \frac{\sin(-x)}{1+(-x)^2} = \frac{-\sin(x)}{1+x^2} = -f(x)$$

$$\therefore I = \int_{-1}^{1} \frac{\sin(x)}{1+x^2}\,\mathrm{d}x = -\int_{-1}^{0} \frac{\sin(x)}{1+x^2}\,\mathrm{d}x + \int_{0}^{1} \frac{\sin(x)}{1+x^2}\,\mathrm{d}x \boxed{= 0}$$

3.6 Fundamental Theorem of Calculus

Take the derivative of $y(x) = \displaystyle\int_{2x}^{3x+1} \sin(t^4)\,\mathrm{d}t$

$$
\begin{aligned}
&= \int_{2x}^{0} \sin(t^4)\,\mathrm{d}t \;+\; \int_{0}^{3x+1} \sin(t^4)\,\mathrm{d}t \\
&= -\int_{0}^{2x} \sin(t^4)\,\mathrm{d}t \;+\; \int_{0}^{3x+1} \sin(t^4)\,\mathrm{d}t
\end{aligned}
$$

Suppose the antiderivative of the integrand is F, then plugging in the limits of integration yields

$$
\begin{aligned}
y(x) &= -F(2x) + F(0) + F(3x+1) - F(0) \\
&= F(3x+1) - F(2x)
\end{aligned}
$$

Differentiating takes F back to the original integrand with the limits of integration as the variable, DO NOT FORGET THE CHAIN RULE.

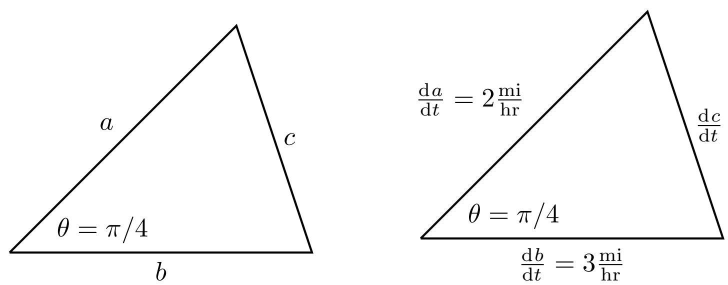
$$
\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \sin((3x+1)^4)\cdot 3 - \sin((2x)^4)\cdot 2 \\
&\boxed{= 3\sin((3x+1)^4) - 2\sin(16x^4)}
\end{aligned}
$$

4 Applications of Derivatives

4.1 Related Rates Part I

Two people start from the same point. One walks east at 3 mi/hr and the other walks northeast 2 mi/hr. How fast is the distance between the people changing after 15 minutes?

Start by drawing a distance triangle and a rate triangle.



Translate the problem to mathematical language: Find $\frac{dc}{dt}$ at $t = \frac{1}{4}hr$

Using Law of Cosines, write an expression that will contain (after differentiating implicitly) $\frac{dc}{dt}$:

$$c^2 = a^2 + b^2 - 2ab\cos(\theta)$$

Differentiate implicitly with respect to t

$$2c\frac{\mathrm{d}c}{\mathrm{d}t} = 2a\frac{\mathrm{d}a}{\mathrm{d}t} + 2b\frac{\mathrm{d}b}{\mathrm{d}t} - 2\cos(\theta)\left(a\frac{\mathrm{d}b}{\mathrm{d}t} + b\frac{\mathrm{d}a}{\mathrm{d}t}\right) \tag{1}$$

After 15 minutes:

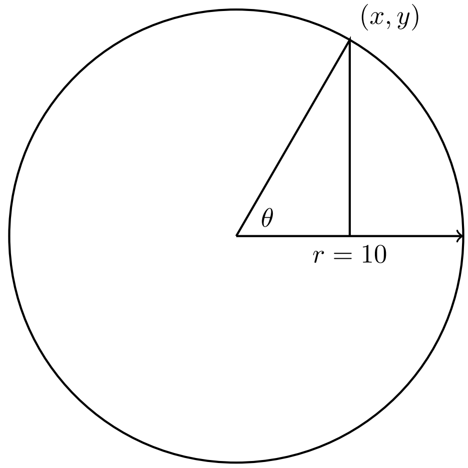
$$\begin{aligned} a &= \frac{1}{4}\cdot 2 = \frac{1}{2}\text{ (mi.)},\ b = \frac{1}{4}\cdot 3 = \frac{3}{4}\text{ (mi.)} \\ c^2 &= \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^2 - 2\frac{1}{2}\frac{3}{4}\cos(\frac{\pi}{4}) \\ c &= \sqrt{\frac{9}{16} + \frac{1}{4} - \frac{3}{4}\left(\frac{\sqrt{2}}{2}\right)} \\ &= \sqrt{\frac{13}{16} - \frac{3\sqrt{2}}{8}} = \sqrt{\frac{1}{16}(13 - 6\sqrt{2})} \end{aligned}$$

Now plug all the values into equation (1):

$$\begin{aligned} \frac{\mathrm{d}c}{\mathrm{d}t} &= \sqrt{\frac{16}{13 - 6\sqrt{2}}}\left(\frac{3}{4}\cdot 3 + \frac{1}{2}\cdot 2 - \frac{\sqrt{2}}{2}\left(\frac{3}{4}\cdot 2 + \frac{1}{2}\cdot 3\right)\right) \\ &= \frac{4}{\sqrt{13 - 6\sqrt{2}}}\left(\frac{9}{4} + 1 + \frac{3\sqrt{2}}{2}\right) \\ &= \frac{1}{\sqrt{13 - 6\sqrt{2}}}\left(13 - 6\sqrt{2}\right) \\ &\boxed{= \sqrt{13 - 6\sqrt{2}}} \end{aligned}$$

4.2 Related Rates Part II

A Ferris wheel with a radius of 10 m is rotating at a rate of one revolution every 2 minutes. How fast is a rider rising when his seat is 16 m above ground level?



$$\omega = \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{1}{2}\frac{rev}{min} = \pi\frac{rad}{min}$$

Since we are looking for $\frac{\mathrm{d}y}{\mathrm{d}t}$, use

$$\tan(\theta) = \frac{y}{x}$$

We have to eliminate the variable x, so use the fact that the point (x, y) is constrained to be on the circle:

$$x = \sqrt{100 - y^2}$$
$$\tan(\theta) = y \cdot (100 - y^2)^{-1/2}$$

Differentiate implicity with respect to t

$$\sec^2(\theta)\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}t}(100 - y^2)^{-1/2} + y\left[-\frac{1}{2}(100 - y^2)^{-3/2} \cdot (-2y\frac{\mathrm{d}y}{\mathrm{d}t})\right]$$
$$\sec^2(\theta)\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}t}\left[(100 - y^2)^{-1/2} + \frac{y^2}{(100 - y^2)^{3/2}}\right]$$

Since we fixed the origin at the center of the circle, the height of the rider is $y = 16 - 10 = 6$, then we can solve for $\cos(\theta)$ by looking at the triangle in the circle:

We see that, $\cos(\theta) = \frac{4}{5}$

Plugging y and $\cos(\theta)$ into our expression we get: $\frac{25}{16}\pi = \frac{\mathrm{d}y}{\mathrm{d}t}\left(\frac{1}{8} + \frac{36}{8^3}\right)$

Simplifying yields, $\boxed{\frac{\mathrm{d}y}{\mathrm{d}t} = 8\pi}$

4.3 Differentials

Use differentials to estimate the $\sin(1°)$.

To use differentials we need to write a function; since the question involves the sine function, use

$$y = \sin(x)$$

Now using the definition of the differential $\mathrm{d}y = f'(x)\mathrm{d}x$, write

$$\mathrm{d}y = \cos(x)\mathrm{d}x$$

Since we are approximating $1°$ which is very close to 0, take x to be 0, and then $\mathrm{d}x = 1° = \frac{\pi}{180}$. Note that we have converted degrees to radians because the normal formulas for the derivatives of trig. functions hold only when the argument is in radians.

$$\mathrm{d}y = \cos(0) \cdot \frac{\pi}{180}$$
$$\boxed{= \frac{\pi}{180} \approx 0.0175}$$

4.4 Linear Approximations

Verify the linear approximation at 0 for the sine function:

$$\sin(x) \approx x$$

Recall that the linearization of f at a is

$$L(x) = f(a) + f'(a)(x - a)$$

Where, in our case, $a = 0$. Thus,

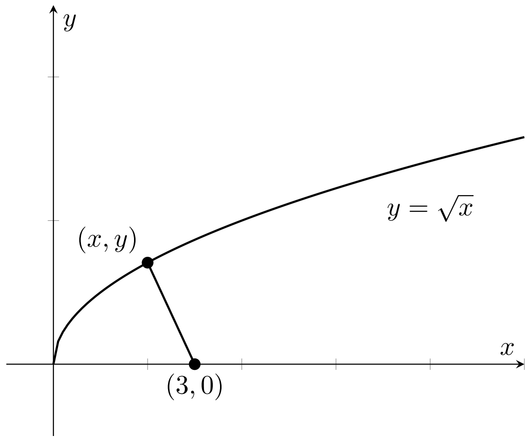
$$\begin{aligned} f(0) &= \sin(0) = 0 \\ f'(0) &= \cos(0) = 1 \\ \longrightarrow L(x) &= 0 + 1 \cdot (x - 0) = x \end{aligned}$$

This tells us that, for small enough x (from 0 to around $20°$), the sine function can be approximated (to within 2% of the actual value) by the linear function $y = x$.

4.5 Optimization (Standard)

Find the point on the curve $y = \sqrt{x}$ that is closest to the point $(3, 0)$.

Begin by drawing a picture:



Choose a general point on the curve and then since we are trying to minimize distance, use this point to write out the distance formula:

$$\begin{aligned} d &= \sqrt{(x-3)^2 + (\sqrt{x} - 0)^2} \\ &= \sqrt{x^2 - 6x + 9 + x} \\ &= \sqrt{x^2 - 5x + 9} \end{aligned}$$

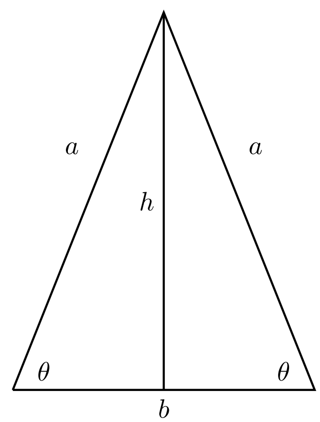
Now take the first derivative and find the critical point(s)

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}(d) &= \frac{2x - 5}{2\sqrt{x^2 - 5x + 9}} = 0 \\ 2x &= 5 \\ x &= \frac{5}{2} \end{aligned}$$

Therefore the point closest to $(3, 0)$ is the point $\left(\frac{5}{2}, \sqrt{\frac{5}{2}}\right)$

4.6 Optimization (Difficult)

Show that of all the isosceles triangles with a given perimeter, the one with the greatest area is equilateral.



The first thing to do with optimization problems is to draw a picture. Then the general procedure is to write out a constraint equation and the equation that you wish to optimize, obviously these equations depend on the geometry of the problem. In our case the triangle is constrained to have a given (i.e. constant) perimeter, say p:

$$p = 2a + b$$

Then the optimization equation is the area of the triangle:

$$A = 2 \cdot \frac{b}{2} \cdot \frac{1}{2} \cdot h = \frac{ab}{2} \sin(\theta)$$

Notice that to prove the triangle is isosceles we need to show that $b = a$. Before we can take the derivative (to find the critical points) we need to eliminate two of the variables. One elimination follows directly from the constrain equation:

$$b = p - 2a$$

Notice that we can eliminate $\sin(\theta)$ because we can find $\cos(\theta)$ entirely in terms of b and a:

$$\begin{aligned} \cos(\theta) &= \frac{b}{2a} \\ \cos^2(\theta) &= \frac{b^2}{4a^2} \\ \sin^2(\theta) = 1 - \cos^2(\theta) &= 1 - \frac{b^2}{4a^2} \\ \sin(\theta) &= \sqrt{1 - \frac{b^2}{4a^2}} \end{aligned}$$

Then,

$$A = \frac{a(p - 2a)}{2} \sqrt{1 - \frac{(p - 2a)^2}{4a^2}}$$

Simplify a little before differentiating

$$\begin{aligned}
&= \frac{a(p-2a)}{2}\sqrt{\frac{4a^2-(p^2-4ap+4a^2)}{4a^2}} \\
&= \frac{a(p-2a)}{4a}\sqrt{4ap-p^2} \\
&= \frac{p-2a}{4}\sqrt{4ap-p^2}
\end{aligned}$$

Now differentiate and equate to zero:

$$\begin{aligned}
\frac{\mathrm{d}A}{\mathrm{d}a} &= \frac{1}{4}\left(-2\sqrt{4ap-p^2}+(p-2a)\frac{4p}{2\sqrt{4ap-p^2}}\right)=0 \\
2\sqrt{4ap-p^2} &= \frac{2p^2-4ap}{\sqrt{4ap-p^2}} \\
4ap-p^2 &= p^2-2ap \\
6ap &= 2p^2 \\
6a &= 2p \\
3a &= p
\end{aligned}$$

Notice that this critical point must correspond to a maximum because physically a minimum would be $A = 0$ which would imply the dimensions are zero. Plugging this value for a back into the equation for b

$$b = 3a - 2a = a$$

∴ The triangle is equilateral because all sides have the same length. □

4.7 Curve Sketching

Sketch the curve $y = 4x - \tan(x)$, $\frac{-\pi}{2} < x < \frac{\pi}{2}$

1. **Domain.** In this case the domain is already specified.
2. **Intercepts.** To find the y-intercept plug in $x = 0$:

$$y(0) = 4(0) - \tan(0) = 0$$

   Thus the curve passes through the origin (i.e. (0,0)).

3. **Symmetry.** Check if y is even or odd:

$$\begin{aligned} y(-x) &= 4(-x) - \tan(-x) \\ &= -4x - \frac{\sin(-x)}{\cos(-x)} \\ &= -4x + \frac{\sin(x)}{\cos(x)} = -4x + \tan(x) \\ y(-x) &\neq y(x) \longrightarrow y \text{ is not even.} \\ y(-x) &= -y(x) \longrightarrow y \text{ is odd.} \end{aligned}$$

4. **Asymptotes.**
   (a) **Horizontal.** Since the domain is restricted there cannot be any horizontal asymptotes.
   (b) **Vertical.**

$$\begin{aligned} &\lim_{x \to a} (4x - \tan(x)) = \pm\infty \\ &y \longrightarrow -\infty \text{ as } x \longrightarrow \frac{\pi}{2} \\ &y \longrightarrow \infty \text{ as } x \longrightarrow \frac{-\pi}{2} \end{aligned}$$

5. **First Derivative.**

$$\begin{aligned} \frac{dy}{dx} &= 4 - \sec^2(x) = 0 \\ 4 &= \sec^2(x) \\ \pm 2 &= \frac{1}{\cos(x)} \longrightarrow \cos(x) = \pm\frac{1}{2} \\ x &= \pm\frac{\pi}{3} \text{ (critical points)} \end{aligned}$$

6. **Second Derivative.**

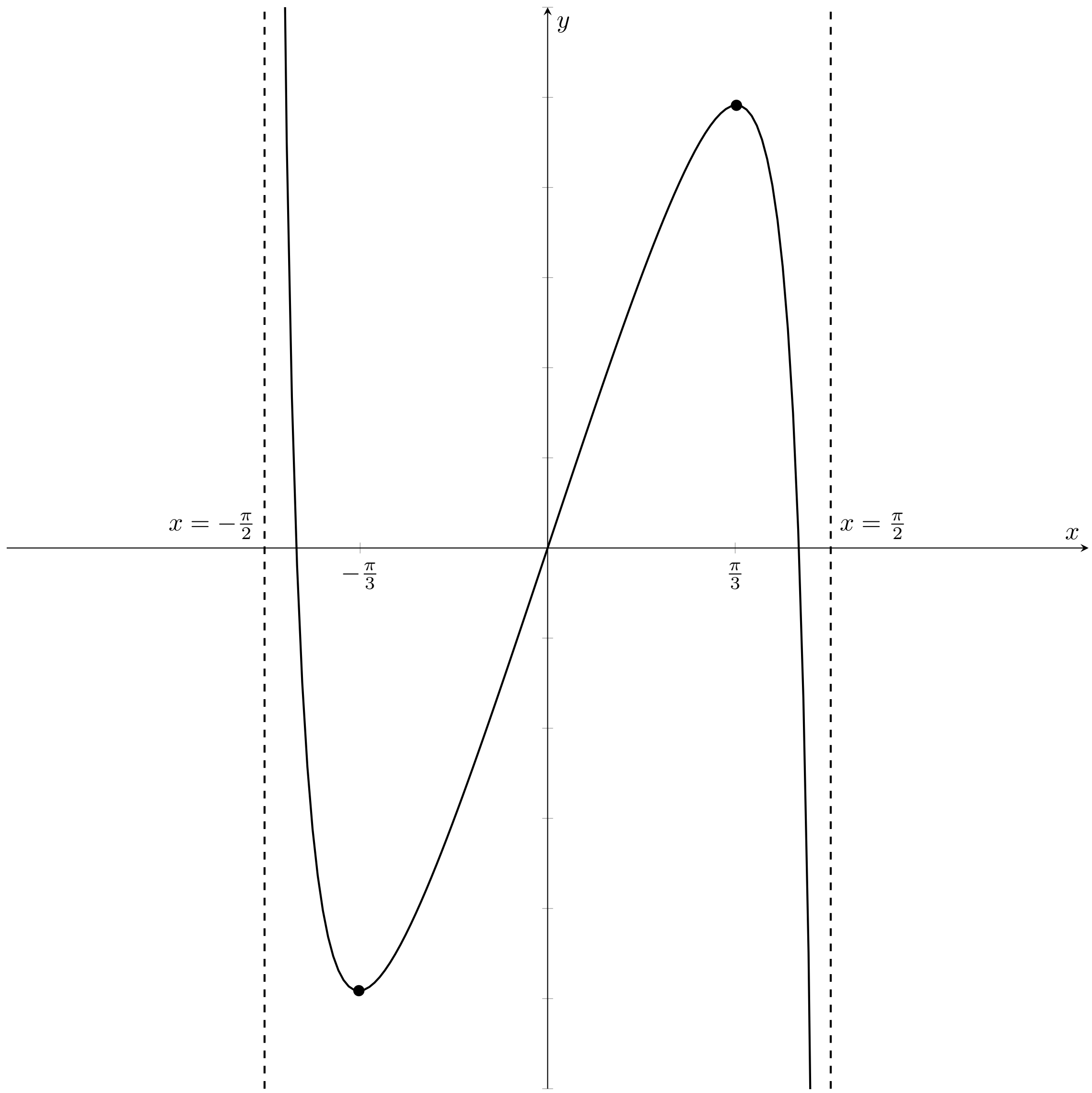
$$\frac{d^2y}{dx^2} = -2\sec^2(x)\tan(x) = 0$$

   Notice that $\sec^2(x)$ never equals zero because of the domain restriction, so you can divide both sides by $-2\sec^2(x)$

$$\begin{aligned} \tan(x) &= 0 \\ x &= 0 \text{ (point of inflection)} \end{aligned}$$

   Test concavity on each side of the inflection point:

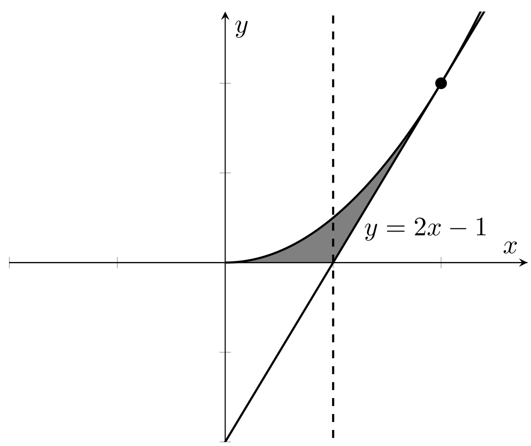
$$\begin{aligned} -\frac{\pi}{2} < x < 0 : \quad & y'' > 0 \Longrightarrow x = -\frac{\pi}{3} \text{ is a local min.} \\ 0 < x < \frac{\pi}{2} : \quad & y'' < 0 \Longrightarrow x = \frac{\pi}{3} \text{ is a local max.} \end{aligned}$$


y
x
$x = -\frac{\pi}{2}$
$x = \frac{\pi}{2}$
$-\frac{\pi}{3}$
$\frac{\pi}{3}$

5 Applications of Integrals

5.1 Area Between Curves

Find the area of the region bounded by the parabola $y = x^2$, the tangent line to this parabola at $(1, 1)$, and the x-axis.



After drawing a picture, the first step is to find the equation of the tangent line.

$$y' = 2x\big|_{x=1} = 2$$

Then using point-slope form,

$$\begin{aligned} y - 1 &= 2(x - 1) \\ y &= 2x - 1 \end{aligned}$$

Notice that to find the area enclosed by the curves, two integrals are needed because the "bottom" curve changes. This point of change occurs when the two "bottom" curves intersect, i.e. when the tangent line hits the x-axis

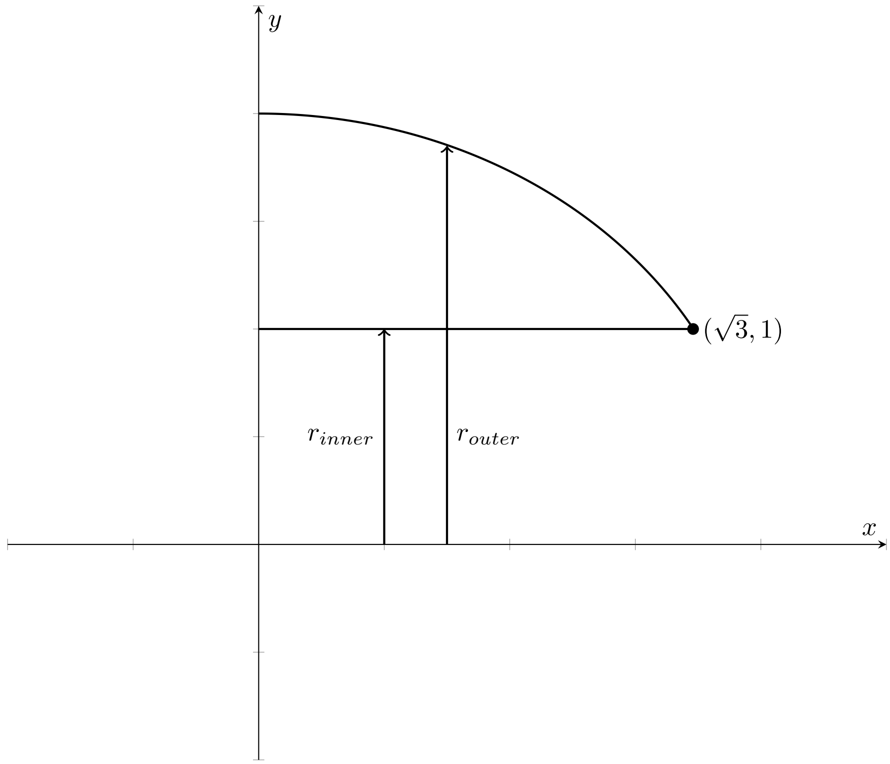
$$\begin{aligned} y &= 0 = 2x - 1 \\ x &= \frac{1}{2} \end{aligned}$$

Then the total area (A) is,

$$
\begin{aligned}
A &= \int_0^{\frac{1}{2}} (x^2 - 0)\,\mathrm{d}x \; + \int_{\frac{1}{2}}^{1} (x^2 - (2x-1))\,\mathrm{d}x \\
&= \left.\frac{x^3}{3}\right|_0^{\frac{1}{2}} \; + \; \left.\left(\frac{x^3}{3} - x^2 + x\right)\right|_{\frac{1}{2}}^{1} \\
&\boxed{= \frac{1}{12}}
\end{aligned}
$$

5.2 Volume of Revolution, Disk Method, Part I

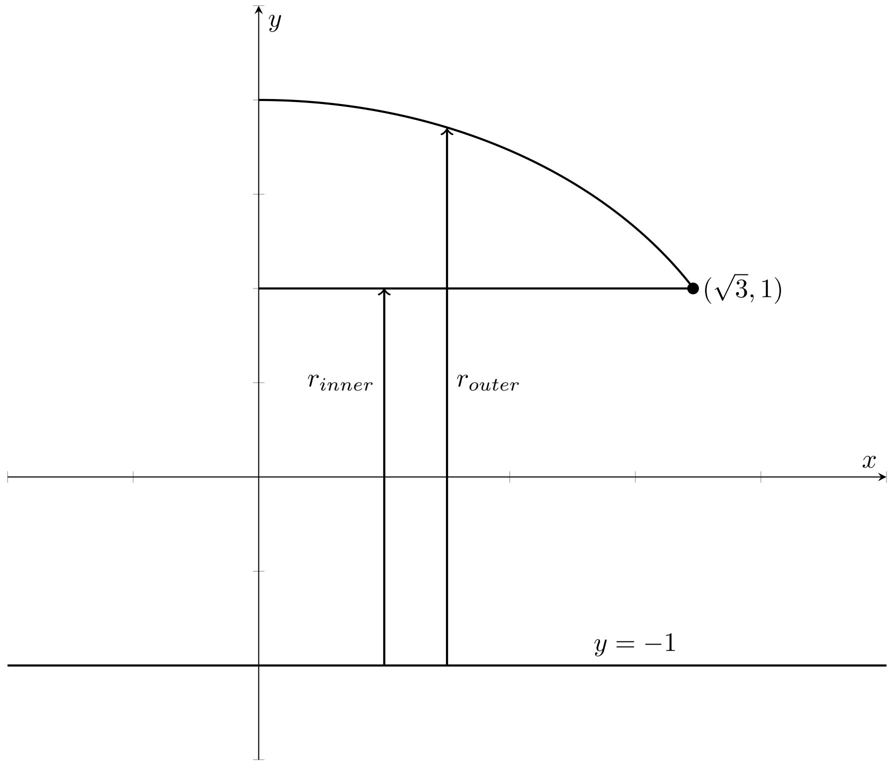
Find the volume of the solid obtained by revolving the area enclosed by $f(x) = \sqrt{4 - x^2}$ and $g(x) = 1$ from $0 \leq x \leq \sqrt{3}$ about the x-axis.



$$\begin{aligned}
\text{Volume} &= \pi \int_a^b (r_{outer}^2 - r_{inner}^2)\,\mathrm{d}x \\
&= \pi \int_0^{\sqrt{3}} (4 - x^2 - 1)\,\mathrm{d}x \\
&= \pi \int_0^{\sqrt{3}} (3 - x^2)\,\mathrm{d}x \\
&= \pi \left(3x - \frac{x^3}{3}\right)\Bigg|_0^{\sqrt{3}} \\
&= \pi \left(3\sqrt{3} - \frac{3\sqrt{3}}{3}\right) \\
&\boxed{= 2\pi\sqrt{3}}
\end{aligned}$$

5.3 Volume of Revolution, Disk Method, Part II

Using the same region enclosed by the functions from Problem 29, set up, but do not evaluate, the integral expression for volume if the axis of revolution is moved to $y = -1$.



From the picture, it is easy to see that both radii have been lengthened by 1 unit:

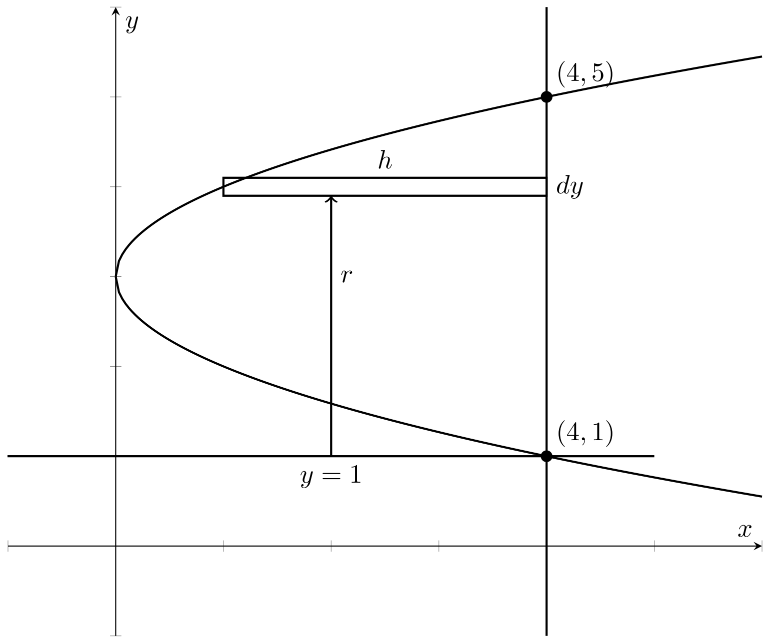
$$r_{outer} = \sqrt{4 - x^2} + 1$$
$$r_{inner} = 1 + 1 = 2$$

Then plugging into the volume equation:

$$\boxed{\text{Volume} = \pi \int_0^{\sqrt{3}} \left[(\sqrt{4 - x^2} + 1)^2 - 4\right] \mathrm{d}x}$$

5.4 Volume of Revolution, Shell Method

Find the volume obtained by revolving the region bounded by the curves $x = (y-3)^2, x = 4$ about $y = 1$.

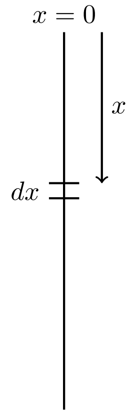


The expression for the height of a general shell will be, $h = 4 - (y-3)^2$
The radius, as seen from the picture, can be expressed as $r = y - 1$
And the limits of integration will be $y : 1 \longrightarrow 5$
Now plugging into the equation for volume via shells we get:

$$\begin{aligned}
\text{Volume} &= 2\pi \int_1^5 (y-1)(4-(y-3)^2)\,\mathrm{d}y \\
&= 2\pi \int_1^5 (y-1)(4-(y^2-6y+9))\,\mathrm{d}y \\
&= 2\pi \int_1^5 (y-1)(-y^2+6y-5)\,\mathrm{d}y \\
&= 2\pi \int_1^5 (-y^3+6y^2-5y+y^2-6y+5)\,\mathrm{d}y \\
&= 2\pi \int_1^5 (-y^3+7y^2-11y+5)\,\mathrm{d}y \\
&= 2\pi \left(-\frac{y^4}{4} + 7\frac{y^3}{3} - 11\frac{y^2}{2} + 5y \right)\Bigg|_1^5 \\
&= 2\pi \left[\left(-\frac{625}{4} + 7\frac{125}{3} - 11\frac{25}{2} + 25 \right) - \left(-\frac{1}{4} + 7\frac{1}{3} - 11\frac{1}{2} + 5 \right) \right] \\
&= 2\pi \left[\left(\frac{275}{12} \right) - \left(\frac{19}{12} \right) \right] \\
&= 2\pi \cdot \frac{256}{12} \\
&\boxed{= \frac{128}{3}\pi}
\end{aligned}$$

5.5 Work, Chain Problem

A cable 50 feet in length and weighing 4 pounds per foot (lb/ft) is hanging. Calculate the work done in winding up 25 feet of the cable. Neglect all forces except gravity.



This problem can be broken into two parts; each little segment dx in the first 25 feet of the cable will move a different length, namely a distance x. Whereas each segment of the bottom 25 feet of the cable will move a constant distance of 25 feet.

Top 25 feet:
The force due to gravity on a small length, dx, of the cable is the linear density times the length: $4\ dx$.

Fixing the origin of the number line at the top of the cable will prove to be the easiest configuration because then each small segment of the cable will move a length x. Then because Work=Force×Distance:

$$\begin{aligned} \text{Work} &= \int_0^{25} 4x \, \mathrm{d}x \\ &= 2x^2 \Big|_0^{25} \\ &= 2 \cdot 625 = 1250 \text{ ft.-lb} \end{aligned}$$

Bottom 25 feet:
Because all segments move a constant distance, an integral is not needed:

$$\text{Work} = (25 \cdot 4) \cdot 25 = 2500 \text{ ft.-lb}$$

$$\boxed{\text{Thus the total work is } 2500 + 1250 = 3750 \text{ ft.-lb}}$$

5.6 Work, Spring Problem

A spring has a natural length of 1 meter (m). A force of 100 Newtons compresses it to 0.9 m. How much work is required to compress it to half of its natural length? What is the length of the spring when 20 Joules of work have been expended?

Here we have to use Hooke's Law, $f(x) = kx$, then work $= \int_a^b kx\,\mathrm{d}x$. Keep in mind that all distances have to be measured with respect to the natural length of the spring, e.g. stretching the spring to a length of 1.5 m means that you have to stretch it 0.5 m (w.r.t. natural length).

The first thing to do is to solve for the constant k:

$$\begin{aligned} 100 &= k(0.1) \\ k &= 1000 \ \ \frac{N}{m} \end{aligned}$$

Compressing the spring to half of the natural length means that the limits of our work integral will be from 0 to 0.5

$$\begin{aligned} f &= 1000x \\ W &= \int_0^{0.5} 1000x\,\mathrm{d}x \\ &= 500x^2\Big|_0^{0.5} \\ &\boxed{= \frac{500}{4} = 125 \text{ J}} \end{aligned}$$

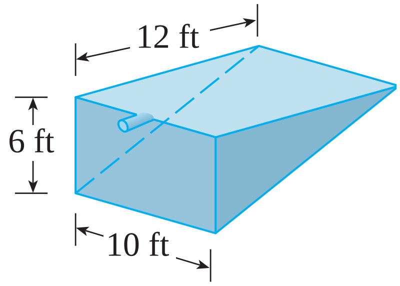
In the second part of the problem we are given that $W = 20$J;

$$\begin{aligned} W = 20 &= \int_0^{a} 1000x\,\mathrm{d}x \\ 20 &= 500x^2\Big|_0^{a} \\ 20 &= 500a^2 \\ a &= \sqrt{\frac{20}{500}} = \sqrt{\frac{1}{25}} = \frac{1}{5} \end{aligned}$$

So if we are still compressing the spring, the length is now 0.8m

5.7 Work, Involving Geometry

If the tank is filled with water, how much work is required to pump all the water out of the top of the tank? Use the fact that the density of water is 62.5 lb./ft^3.

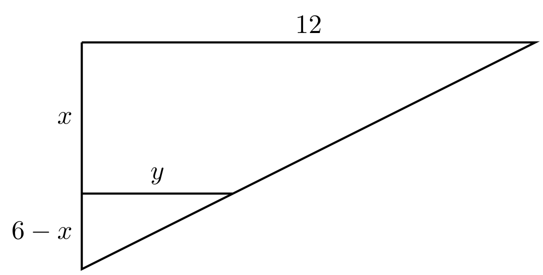


$$force = (mass) \cdot (gravity) = \rho V g$$

where ρ is the density of water and V is the volume. Then if we let $x = 0$ at the top of the tank, then each each slice of the tank is going to have to be lifted a distance of x, so the total work becomes:

$$Work = \rho g \int_a^b V(x) \cdot x \, dx$$

To come up with an expression for how the volume of a slice of the tank varies with height we can use similar triangles. Since the front face of the tank has a constant width, look at the tank in profile:



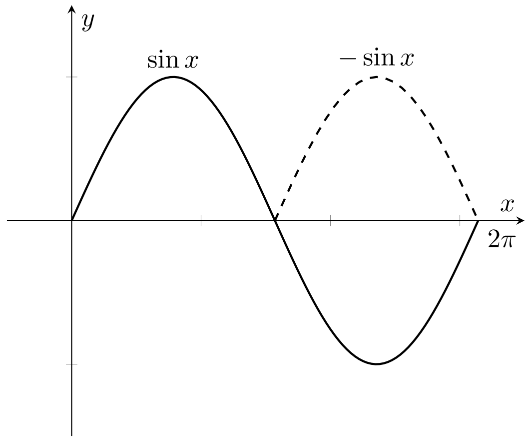
$$\frac{12}{6} = 2 = \frac{y}{6-x}$$
$$y = 2(6-x)$$
$$V(x) = 10y \text{ d}x = 20(6-x) \text{ d}x$$

Then the work becomes,

$$\text{Work} = \rho g \int_0^6 20(6-x)x \, dx = 20\rho g \int_0^6 (6x - x^2) \, dx$$
$$= 20\rho g \left(3x^2 - \frac{x^3}{3}\right)\Big|_0^6$$
$$= 20\rho g \left(108 - \frac{216}{3}\right)$$
$$= 20(62.5)(36)$$
$$\boxed{= 45000 \text{ ft.-lb}}$$

5.8 Average Value of a Function

Find the average value of $f(x) = |\sin(x)|$ from $0 \leq x \leq 2\pi$.



$$\text{Average value} = \frac{1}{2\pi - 0}\int_0^{2\pi} |\sin(x)|\,\mathrm{d}x$$

$$f(x) = \begin{cases} \sin(x) & \text{if } 0 < x < \pi \\ -\sin(x) & \text{if } \pi < x < 2\pi \end{cases}$$

$$= \frac{1}{2\pi}\left(\int_0^{\pi} \sin(x)\,\mathrm{d}x \ + \int_{\pi}^{2\pi} -\sin(x)\,\mathrm{d}x\right)$$

$$= \frac{1}{2\pi}\left(-\cos(x)\Big|_0^{\pi} + \cos(x)\Big|_{\pi}^{2\pi}\right)$$

$$= \frac{1}{2\pi}(1+1+1+1) = \frac{4}{2\pi}$$

$$\boxed{= \frac{2}{\pi}}$$