# Contents

# 1 Real Numbers, Exponents, and Radicals

## 1.1 Rationalizing the Denominator

Simplify and rationalize the denominator when appropriate.

$$\sqrt[4]{\frac{5x^8y^3}{27x^2}}$$

## Solution

First simplify the expression

$$\sqrt[4]{\frac{5\cancel{x^2}x^6y^3}{3^3\cancel{x^2}}} = \sqrt[4]{\frac{5x^6y^3}{3^3}}$$

Rationalize the denominator so that it only contains exponents that are multiples of 4

$$\frac{\sqrt[4]{5x^6y^3}}{\sqrt[4]{3^3}} \cdot \frac{\sqrt[4]{3}}{\sqrt[4]{3}} = \sqrt[4]{\frac{15x^6y^3}{3^4}}$$

Simply both the denominator and numerator

$$\sqrt[4]{\frac{15x^6y^3}{3^4}} = \frac{x}{3}\sqrt[4]{15x^2y^3}$$

## 1.2 Factoring Polynomials

(a) $64x^3 - y^6$

(b) $y^2 - x^2 + 8y + 16$

## Solution

(a) Recognize this as a difference of cubes

$$(4x)^3 - (y^2)^3$$

Use the difference of cubes formula to factor

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$
$$(4x)^3 - (y^2)^3 = (4x - y^2)(16x^2 + 4xy^2 + y^4)$$

(b) Rearrange to form a perfect square

$$y^2 + 8y + 16 - x^2$$

$$y^2 + 8y + 16 - x^2 = (y+4)^2 - x^2$$

Recognize this as a difference of squares

$$a^2 - b^2 = (a+b)(a-b)$$
$$(y+4)^2 - x^2 = (y+4+x)(y+4-x)$$

### 1.3 Algebraic and Fractional Expressions

Simplify the following expression

$$\frac{(4x^2+9)^{1/2}(2)-(2x+3)\left(\frac{1}{2}\right)(4x^2+9)^{-1/2}(8x)}{[(4x^2+9)^{1/2}]^2}$$

## Solution

Begin by factoring out the least common multiple, in this case $(4x^2+9)^{-1/2}$.

$$\frac{(4x^2+9)^{-1/2}\left[(4x^2+9)(2)-(2x+3)\left(\frac{1}{2}\right)(8x)\right]}{\left[\ (4x^2+9)^{1/2}\right]^2}$$

Then reduce exponents

$$\frac{(4x^2+9)(2)-(2x+3)\left(\frac{1}{2}\right)(8x)}{(4x^2+9)^{3/2}}$$

Finally distribute and combine terms in the numerator

$$\begin{aligned}
&\frac{8x^2+18-(8x^2+12x)}{(4x^2+9)^{3/2}}\\
&=\frac{\cancel{8x^2}+18\cancel{-8x^2}-12x}{(4x^2+9)^{3/2}}\\
&=\frac{18-12x}{(4x^2+9)^{3/2}}\\
&=\frac{6(3-2x)}{(4x^2+9)^{3/2}}
\end{aligned}$$

## 1.4 Equations

(a) Solve for the specified variable

$$S = \frac{p}{q + p(1-q)} \quad \text{for } q$$

(b) Solve the equation

$$\frac{2}{2x+1} - \frac{3}{2x-1} = \frac{-2x+7}{4x^2-1}$$

(c) Solve for $x$

$$x = 4 + \sqrt{4x-19}$$

## Solution

(a) Begin by multiplying the denominator over to the left hand side

$$\begin{aligned} S &= \frac{p}{q + p(1-q)} \\ S(q + p(1-q)) &= p \end{aligned}$$

Distribute $S$ and $p$

$$\begin{aligned} S(q + p(1-q)) &= p \\ Sp + Sq - Spq &= p \end{aligned}$$

Isolate $q$ and solve

$$\begin{aligned} Sp + Sq - Spq &= p \\ Sq - Spq &= p - Sp \\ q(S - Sp) &= p - Sp \\ q &= \frac{p - Sp}{S - Sp} \\ q &= \frac{p(1-S)}{S(1-p)} \end{aligned}$$

(b) First find the Least Common Denominator (LCD) and then multiply the entire equation by it

$$\frac{2}{2x+1} - \frac{3}{2x-1} = \frac{-2x+7}{(2x-1)(2x+1)}$$

$$\left(\frac{2}{2x+1} - \frac{3}{2x-1} = \frac{-2x+7}{(2x-1)(2x+1)}\right)(2x-1)(2x+1)$$

$$\frac{2(2x-1)\cancel{(2x+1)}}{\cancel{2x+1}} - \frac{3\cancel{(2x-1)}(2x+1)}{\cancel{2x-1}} = \frac{-2x+7\cancel{(2x-1)(2x+1)}}{\cancel{(2x-1)(2x+1)}}$$

$$2(2x-1) - 3(2x+1) = -2x+7$$

Next combine like terms and solve for $x$

$$\begin{aligned} 4x - 2 - 6x - 3 &= -2x + 7 \\ 4x + 2x - 6x &= 7 + 3 + 2 \\ 0 &\neq 12 \end{aligned}$$

Since zero is not equal to twelve the equation has **no solutions**.

(c) Begin by isolating the square root

$$\begin{aligned} x &= 4 + \sqrt{4x - 19} \\ x - 4 &= \sqrt{4x - 19} \end{aligned}$$

Square both sides to eliminate the square root and F.O.I.L

$$\begin{aligned} (x-4)^2 &= (\sqrt{4x-19})^2 \\ x^2 - 8x + 16 &= 4x - 19 \end{aligned}$$

Move everything to the left hand side and combine terms

$$\begin{aligned} x^2 - 8x - 4x + 16 + 19 &= 0 \\ x^2 - 12x + 35 &= 0 \end{aligned}$$

Factor and solve for $x$

$$(x-7)(x-5) = 0$$

$$\begin{aligned} (x-7) &= 0 \quad &\text{or} \quad && (x-5) &= 0 \\ x &= 7 \quad &\text{or} \quad && x &= 5 \end{aligned}$$

**NOTE:** Don't forget to check your answers when squaring both sides. In this particular problem, $x = 5$ and $x = 7$ are both solutions to the equation.

### 1.5 Applied Problems

In a certain medical test designed to measure carbohydrate tolerance, an adult drinks 7 ounces of a 30% glucose solution. When the test is administered to a child, the glucose concentration must be decreased to 20%. How much 30% glucose solution and how much water should be used to prepare 7 ounces of 20% glucose solution?

## Solution

In order to solve this problem, it is necessary to visualize the problem first.

| Children Solution | = | Adult Solution | + | Water |
|---|---|---|---|---|
| 20% glucose | = | 30% glucose | + | 0% glucose |
| 7 ounces | = | x | + | (7-x) ounces |

From the table above, set up the equation.

$$(0.20)(7) = (0.30)(x) + (0.00)(7 - x)$$

Simplify the equation

$$1.4 = 0.3x$$

Solve for $x$

$$x = \frac{1.4}{0.3}$$

$$x = \frac{14}{3}$$

$\frac{14}{3}$ is the amount of 30% solution needed to prepare 7 ounces of 20% glucose solution.

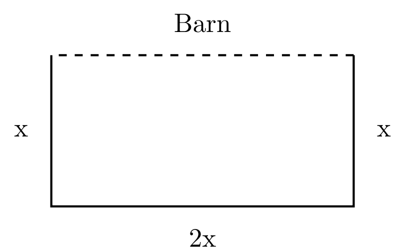
Now calculate the water amount needed.

$$\begin{aligned} \text{water} &= 7 - x \\ &= 7 - \frac{14}{3} \\ &= \frac{21}{3} - \frac{14}{3} \\ \text{water} &= \frac{7}{3} \text{ ounces} \end{aligned}$$

A farmer plans to close a rectangular region, using part of his barn for one side and fencing for the other three sides. If the side parallel to the barn is to be twice the length of the adjacent side, and the area of the region is to be $128ft^2$, how many feet of fencing should be purchased?

## Solution

For most word problems, it is important to visualize the problem/situation.




$$\begin{aligned} \text{Area of rectangle} &= (2x)(x) \\ \text{Perimeter of rectangle} &= 2x + 2x \end{aligned}$$

Now that you have a visual, set up the equation.

$$(2x)(x) = 128$$

Simplify the equation and solve for $x$.

$$\begin{aligned} 2x^2 &= 128 \\ x^2 &= \frac{128}{2} \\ x^2 &= 64 \\ \sqrt{x^2} &= \sqrt{64} \\ x &= \pm 8 \end{aligned}$$

Since distances can only be positive, exclude the negative answer. Once you have calculated the width $x$, calculate the perimeter as follows. Note that the barn is acting as one of the sides and does not need to be accounted for:

$$\begin{aligned} \text{Perimeter} &= 2x + x + x \\ &= 4x \\ &= 4(8) \\ \text{Perimeter} &= 32 \text{ ft.} \end{aligned}$$

# 2 Quadratic Equations and Complex Numbers

## 2.1 Quadratic Equations

(a) Solve by completing the square

$$4x^2 - 12x - 11 = 0$$

(b) Solve the equation

$$\frac{3}{2}z^2 - 4z - 1 = 0$$

## Solution

(a) Begin by dividing equation by 4 in order to get a coefficient of 1 in front of $x^2$

$$4x^2 - 12x - 11 = 0$$
$$x^2 - 3x - \frac{11}{4} = 0$$

Add $\frac{11}{4}$ to both sides:

$$x^2 - 3x - \cancel{\frac{11}{4}} + \cancel{\frac{11}{4}} = \frac{11}{4}$$

Complete the square as follows:

$$x^2 - 3x + \left(\frac{b}{2}\right)^2 = \frac{11}{4} + \left(\frac{b}{2}\right)^2$$
$$x^2 - 3x + \left(\frac{3}{2}\right)^2 = \frac{11}{4} + \left(\frac{3}{2}\right)^2$$
$$x^2 - 3x + \frac{9}{4} = \frac{11}{4} + \frac{9}{4}$$
$$\left(x - \frac{3}{2}\right)^2 = \frac{20}{4}$$
$$\left(x - \frac{3}{2}\right)^2 = 5$$

We can solve for $x$ by taking the square root of both sides:

$$\sqrt{\left(x - \frac{3}{2}\right)^2} = \pm\sqrt{5}$$
$$x - \frac{3}{2} = \pm\sqrt{5}$$

Finally, Add $\frac{3}{2}$ to both sides to isolate $x$:

$$x - \cancel{\frac{3}{2}} + \cancel{\frac{3}{2}} = \frac{3}{2} \pm \sqrt{5}$$
$$x = \frac{3}{2} \pm \sqrt{5}$$

(b) We can solve this problem by either completing the square or quadratic formula.

**Remember:** Quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Begin the problem by multiplying by 2 to get rid of the fraction

$$2\left(\frac{3}{2}z^2 - 4z - 1\right) = (2)(0)$$
$$3z^2 - 8z - 2 = 0$$

We will solve for $x$ using the quadratic formula

$$a = 3 \qquad b = -8 \qquad c = -2$$

$$z = \frac{8 \pm \sqrt{8^2 - 4(3)(-2)}}{2(3)}$$

$$= \frac{8 \pm \sqrt{64 + 24}}{6}$$
$$= \frac{8 \pm \sqrt{88}}{6}$$

Try to simplify the radical

$$z = \frac{8 \pm \sqrt{88}}{6}$$

$$= \frac{8 \pm \sqrt{4 \cdot 22}}{6}$$

$$= \frac{8 \pm 2\sqrt{22}}{6}$$

Thus:

$$z = \frac{4}{3} \pm \frac{\sqrt{22}}{3}$$

## 2.2 Complex Numbers

(a) Write the following expression in the form $a + bi$, where $a$ and $b$ are real numbers.

$$\frac{-4+6i}{2+7i}$$

(b) Find the values of $x$ and $y$, where $x$ and $y$ are real numbers.

$$(2x - y) - 16i = 10 + 4yi$$

(c) Find the solutions to the equation

$$4x^4 + 25x^2 + 36 = 0$$

## Solution

(a) When simplifying fractions, it is important to always rationalize the denominator. Begin the problem by multiplying by the conjugate of the denominator and F.O.I.L.

$$\frac{-4+6i}{2+7i} \cdot \frac{2-7i}{2-7i}$$

$$= \frac{(-4+6i)(2-7i)}{(2+7i)(2-7i)}$$

$$= \frac{-8+28i+12i-42i^2}{4-49i^2}$$

Simplify numerator and denominator.

**Remember:**

| | |
|---|---|
| $i$ | $\sqrt{-1}$ |
| $i^2$ | $-1$ |
| $i^3$ | $-\sqrt{-1}$ |
| $i^4$ | $1$ |

$$\frac{-8+28i+12i-42(-1)}{4-49(-1)}$$

$$= \frac{-8+28i+12i+42}{4+49}$$

$$= \frac{34+40i}{53}$$

Write in $a \pm bi$ form.

$$\frac{34}{53} + \frac{40}{53}i$$

(b) The easiest way to solve this problem is by equate the real and imaginary parts

$$(2x - y) - 16i = 10 + 4yi$$

$$\begin{cases} (2x - y) = 10 & (1) \\ -16i = 4yi & (2) \end{cases}$$

First solve for $y$ in equation (2).

$$\begin{aligned} -16\cancel{i} &= 4y\cancel{i} \\ -16 &= 4y \\ y &= -4 \end{aligned}$$

Substitute $y$ in equation (1) and solve for $x$.

$$\begin{aligned} 2x - y &= 10 \\ 2x - (-4) &= 10 \\ 2x + 4 &= 10 \\ 2x &= 10 - 4 \\ x &= \frac{6}{2} \\ x &= 3 \end{aligned}$$

The values of $x$ and $y$ are 3 and $-4$ , respectively.

(c) In order to solve this problem it is important to recognize it as a quadratic problem.
Start by letting $u = x^2$:

$$\begin{aligned} 4x^4 + 25x^2 + 36 &= 0 \\ 4u^2 + 25u + 36 &= 0 \end{aligned}$$

From here we can factor by using $AC$-method (a.k.a factor by grouping) or quadratic formula (we will use the $AC$-method here):

$$\begin{aligned} 4u^2 + 25u + 36 &= 0 \\ 4u^2 + 16u + 9u + 36 &= 0 \\ 4u(u + 4) + 9(u + 4) &= 0 \\ (4u + 9)(u + 4) &= 0 \end{aligned}$$

Set both $u$ factors equal to zero and solve for $u$:

$$\begin{aligned} (4u + 9) &= 0 & \qquad (u + 4) &= 0 \\ u &= \frac{-9}{4} & u &= -4 \end{aligned}$$

Substitute $x^2$ back in for $u$:

$$\begin{aligned} u &= \frac{-9}{4} & \qquad u &= -4 \\ x^2 &= \frac{-9}{4} & x^2 &= -4 \end{aligned}$$

Solve for $x$ by taking the square root of both sides:

$$\sqrt{x^2} = \pm\sqrt{\frac{-9}{4}} \qquad\qquad \sqrt{x^2} = \pm\sqrt{-4}$$
$$x = \pm\frac{3}{2}i \qquad\qquad x = \pm 2i$$

Finally, write into $a \pm bi$ form:

$$x = 0 \pm \frac{3}{2}i \qquad\qquad x = 0 \pm 2i$$

## 2.3 Applied Problems

A baseball is thrown straight upward with an initial speed of 64 $\frac{\text{ft.}}{\text{sec}}$. The number of feet $s$ above the ground after $t$ seconds is given by the equation

$$s = -16t^2 + 64t$$

(i) When will the baseball be 48 feet above the ground?

(ii) When will it hit the ground?

## Solution

(i) To determine when the baseball will be at a height of 48 feet, plug $s = 48$ into the given equation

$$\begin{aligned} 48 &= -16t^2 + 64t \\ 0 &= 16t^2 - 64t + 48 \\ 0 &= 16(t^2 - 4t + 3) \\ 0 &= 16(t-3)(t-1) \\ \implies t &= 3, 1 \end{aligned}$$

(ii) Ground level corresponds to $s = 0$, plugging that in yields:

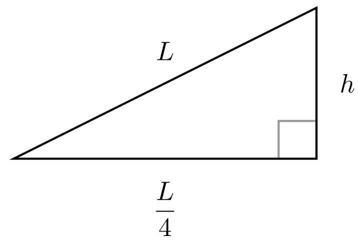
$$\begin{aligned} -16t^2 + 64t &= 0 \\ -16(t^2 - 4t) &= 0 \\ \implies t &= 0, 4 \end{aligned}$$

The $t = 0$ case corresponds to the point right before the ball was thrown, hence the answer we are interested in is $t = 4$.

The recommended distance $d$ that a ladder should be placed away from a vertical wall is 25% of its length $L$. Approximate the height $h$ that can be reached by relating $h$ as a percentage of $L$.

## Solution

In this scenario, it is useful to get a picture of what is going on. Draw a right triangle where the base is $L/4$, the hypotenuse is $L$, and the vertical leg is $h$ as follows:




$$\begin{aligned}\frac{L^2}{16} + h^2 &= L^2 \\ h^2 &= \frac{15L^2}{16} \\ h &= \frac{\sqrt{15}L}{4} \approx .968L\end{aligned}$$

Thus, the approximate height is 97% of $L$

## 2.4 Other Types of Equations

Solve the equation

$$2x^{-\frac{2}{3}} - 7x^{-\frac{1}{3}} - 15 = 0$$

## Solution

In order to begin this problem, we need to use $u$-substitution. Begin by substituting $u$ for $x^{-\frac{1}{3}}$:

$$2x^{-\frac{2}{3}} - 7x^{-\frac{1}{3}} - 15 = 0$$
$$2u^2 - 7u - 15 = 0$$

Next solve for $u$ using your favorite method. We will be factoring and solving for $u$ by setting both factors equal to zero:

$$(2u + 3)(u - 5) = 0$$

$$2u + 3 = 0 \qquad\qquad u - 5 = 0$$
$$u = -\frac{3}{2} \qquad\qquad u = 5$$

Now we can substitute $x^{-\frac{1}{3}}$ back in for $u$:

$$x^{-\frac{1}{3}} = -\frac{3}{2} \qquad\qquad x^{-\frac{1}{3}} = 5$$

Finally, we can solve for $x$ by raising both sides to the negative third power:

$$(x^{-\frac{1}{3}})^{-3} = \left(-\frac{3}{2}\right)^{-3} \qquad\qquad (x^{-\frac{1}{3}})^{-3} = (5)^{-3}$$

Remember that negative exponents, flip coefficients and variables to either the top or bottom and become positive exponents:

$$x = \left(-\frac{2}{3}\right)^{3} \qquad\qquad x = \left(\frac{1}{5}\right)^{3}$$

$$x = -\frac{8}{27} \qquad\qquad x = \frac{1}{125}$$

# 3 Inequalities

## 3.1 Absolute Values

Solve the equation for $x$.

$$3|x+1|-2=-11$$

### Solution

Begin by isolating the absolute value as follows:

$$3|x+1|-2=-11$$

$$3|x+1|=-11+2$$

$$3|x+1|=-9$$

$$|x+1|=-3$$

Since an absolute value will not produce non-negative values as solutions, this problem has **no solution**.

### 3.2 Inequalities and Intervals

Solve and express the solutions in terms of intervals whenever possible.

(a) $-\frac{1}{3}|6-5x|+2 \geq -1$

(b) $\frac{3}{|5-2x|} < 2$

(c) $\frac{x+1}{2x-3} > 2$

## Solution

(a) First, isolate the absolute value:

$$\begin{aligned} -\frac{1}{3}|6-5x|+2 &\geq -1 \\ -\frac{1}{3}|6-5x| &\geq -1 \\ (\cancel{-3})\frac{1}{\cancel{-3}}|6-5x| &\leq -3(-3) \\ |6-5x| &\leq 9 \end{aligned}$$

**Remember**, when you multiply or divide by a negative value, the inequality sign switches directions

Next, set the absolute value to both its positive and negative solutions and solve for $x$:

$$\begin{aligned} 6-5x &\leq 9 \\ \cancel{6}-5x\cancel{-6} &\leq 9-6 \\ -5x &\leq 3 \\ \left(\frac{1}{\cancel{-5}}\right)\cancel{-5}x &\geq 3\left(\frac{1}{-5}\right) \\ x &\geq -\frac{3}{5} \end{aligned} \qquad \begin{aligned} 6-5x &\geq -9 \\ \cancel{6}-5x\cancel{-6} &\geq -9-6 \\ -5x &\geq -15 \\ \left(\frac{1}{\cancel{-5}}\right)\cancel{-5}x &\leq -15\left(\frac{1}{-5}\right) \\ x &\leq 3 \end{aligned}$$

Finally, write solution on interval form:

$$\left[-\frac{3}{5}, 3\right]$$

(b) Unlike other inequalities, we begin this problem by multiplying $|5-2x|$ over to the right hand side. Since $|5-2x|$ will be a nonnegative value, this is allowed and will not change the direction of the inequality:

$$\begin{aligned} \frac{3}{|5-2x|} &< 2 \\ 3 &< 2|5-2x| \end{aligned}$$

Next, we divide by 2 in order to isolate the absolute value:

$$\begin{aligned} \frac{3}{2} &< \frac{\cancel{2}}{\cancel{2}}|5-2x| \\ \frac{3}{2} &< |5-2x| \end{aligned}$$

Next, set the absolute value to both its positive and negative solutions and solve for $x$:

$$\begin{aligned} \frac{3}{2} &< 5 - 2x \\ -2x &> \frac{3}{2} - 5 \\ -2x &> -\frac{7}{2} \\ \frac{\cancel{-2}}{\cancel{-2}}x &< \left(-\frac{7}{2}\right)\left(-\frac{1}{2}\right) \\ x &< \frac{7}{4} \end{aligned} \qquad\qquad \begin{aligned} -\frac{3}{2} &> 5 - 2x \\ -2x &< \frac{3}{2} - 5 \\ -2x &< -\frac{13}{2} \\ \frac{\cancel{-2}}{\cancel{-2}}x &> \left(-\frac{13}{2}\right)\left(-\frac{1}{2}\right) \\ x &> \frac{13}{4} \end{aligned}$$

Finally, write solution on interval form:

$$\left(-\infty, \frac{7}{4}\right) \cup \left(\frac{13}{4}, \infty\right)$$

(c) Begin by moving everything to one side of the inequality to produce a 0 on one of the sides:

$$\frac{x+1}{2x-3} - 2 > 0$$

Next, combine the terms by finding a common denominator as follows:

$$\begin{aligned} \frac{x+1}{2x-3} - \frac{2(2x-3)}{2x-3} &> 0 \\ \frac{x+1-4x+6}{2x-3} &> 0 \\ \frac{-3x+7}{2x-3} &> 0 \end{aligned}$$

We can determine the solution here using a sign chart. However, to do so we must first determine the zeros of both the denominator and numerator:



$$\begin{aligned} -3x + 7 &= 0 \\ -3x &= -7 \\ x &= \frac{7}{3} \end{aligned} \qquad\qquad \begin{aligned} 2x - 3 &= 0 \\ 2x &= 3 \\ x &= \frac{3}{2} \end{aligned}$$

Now we can complete the sign chart as follows:

| | | $\frac{3}{2}$ | $\frac{7}{3}$ |
|---|---|---|---|
| $-3x+7$ | + | + | - |
| $2x-3$ | - | + | + |
| $\frac{-3x+7}{(2x-3)}$ | - | + | - |

From this, we get our answer to be

$$\left(\frac{3}{2}, \frac{7}{3}\right)$$

# 4 Functions and Graphs

## 4.1 Mid-Point

Find a general form of an equation for the perpendicular bisector of a segment $AB$

$$A(3,-1) \qquad B(-2,6)$$

## Solution

First, we need to find the slope od the segment $AB$ using the slope formula.

$$m = \frac{\triangle y}{\triangle x} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\Rightarrow \frac{6-(-1)}{-2-3} = -\frac{7}{5}$$

Next we need to find the slope of the perpendicular bisector. Remember:

$$\text{Perpendicular slope} = -\frac{1}{m}$$
$$\text{Parallel slope} = m$$

The slope of the perpendicular bisector is $\frac{5}{7}$. A perpendicular bisector is defined as a line segment that is both perpendicular to a side and and passes through its midpoint, for this reason we need to find the mid-point of segment $AB$ next.

$$\text{Mid-Point} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$\Rightarrow \left(\frac{-2+3}{2}, \frac{6-1}{2}\right)$$

$$\Rightarrow \left(\frac{1}{2}, \frac{5}{2}\right)$$

Finally, we need to put all the pieces together in order to find the equation of the line. We will begin by first setting our equation in *point − slope*.

$$(y - y_0) = m(x - x_0)$$

$$\left(y - \frac{5}{2}\right) = \frac{5}{7}\left(x - \frac{1}{2}\right)$$

Finally, rearrange into *General Form.*

$$Ax + By = C$$

$$7\left(y - \frac{5}{2}\right) = \frac{5}{\cancel{7}}\left(x - \frac{1}{2}\right)\cancel{7}$$

$$7y - \frac{35}{2} = 5x - \frac{5}{7}$$

$$7y - 5x = -\frac{5}{7} + \frac{35}{2}$$

$$7y - 5x = \frac{30}{2}$$

$$-1\left(7y - 5x = 15\right)$$

$$5x - 7y = -15$$

## 4.2 Circles

(a) Find an equation of the circle that satisfies the stated conditions.

End points of a diameter $A(4, -3)$ and $B(-2, 7)$

### Solution

We begin by realizing that the center of any circle is the midpoint of a diameter, so the center of the circle can be determined as follows.

$$\left(\frac{4+(-2)}{2}, \frac{(-3)+7}{2}\right) = (1, 2)$$

To find the length of the diameter of the circle we use the distance formula.

$$D = \sqrt{(4-(-2))^2 + ((-3)-7)^2} = \sqrt{136} = 2\sqrt{34}$$

Then the radius of the circle is one half of the diameter, so $r = \frac{D}{2} = \sqrt{34}$. Using the general equation of a circle, we have that the equation we need is

$$(x-1)^2 + (y-2)^2 = 34$$

(b) Find the center and radius of the circle with the given equation

$$2x^2 + 2y^2 - 12x + 4y - 15 = 0$$

### Solution

We begin by dividing by 2 and then simplifying further by separating the $x$ terms and the $y$ terms and also moving the constant to the right hand side.

$$\begin{aligned} 2x^2 + 2y^2 - 12x + 4y - 15 &= 0 \\ x^2 + y^2 - 6x + 2y - \frac{15}{2} &= 0 \\ x^2 + y^2 - 6x + 2y &= \frac{15}{2} \\ x^2 - 6x + y^2 + 2y &= \frac{15}{2} \end{aligned}$$

Then we complete the square to put the equation in the standard form of a circle.

$$\begin{aligned} (x^2 - 6x + 9) + (y^2 + 2y + 1) &= \frac{15}{2} + 9 + 1 \\ (x-3)^2 + (y+1)^2 &= \frac{35}{2} \end{aligned}$$

From here we have that the center of the circle is at $(3, -1)$ and that the radius of the circle is given by

$$r = \sqrt{\frac{35}{2}} = \frac{\sqrt{70}}{2}$$

## 4.3 Piecewise Functions

Find the domain and sketch the graph of

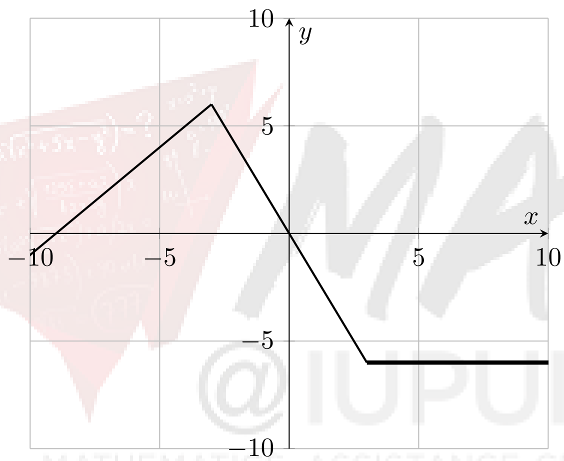
$$f(x) = \begin{cases} x+9 & \text{if } x < -3 \\ -2x & \text{if } |x| \leq 3 \\ -6 & \text{if } x > 3 \end{cases}$$

## Solution

The domain of the function is all numbers $x$ for which $f(x)$ is defined. Since we may plug any number into the function, the domain is given by

$$(-\infty, \infty)$$

To begin graphing the function, we graph it on the three intervals $(-\infty, -3)$, $[-3, 3]$ and $(3, \infty)$. Since they are all lines on each of these intervals, we arrive at the following graph.



### 4.4 Inequality

Solve and express the solution in terms of intervals whenever possible.

$$\frac{x-2}{x^2-3x-10} \geq 0$$

## Solution

Begin by factoring the denominator.

$$\frac{x-2}{x^2-3x-10} = \frac{x-2}{(x-5)(x+2)}$$

From here, make a sign chart to determine where the function is positive or negative.

| | | -2 | | 2 | | 5 | |
|---|---|---|---|---|---|---|---|
| $x-2$ | - | | - | | + | | + |
| $x+2$ | - | | + | | + | | + |
| $x-5$ | - | | - | | - | | + |
| $\frac{x-2}{(x+2)(x-5)}$ | - | | + | | - | | + |

Thus, the solution set is:

$$(-2, 2] \cup (5, \infty)$$

### 4.5 Difference Quotient

Simplify the difference quotient.

$$\frac{f(x+h)-f(x)}{h} \quad \text{if} \quad h \neq 0$$

$$\text{where } f(x) = x^2 + 5$$

## Solution

Begin by plugging in values $x$ and $x+h$ for the function and then simplifying.

$$\frac{f(x+h)-f(x)}{h} = \frac{((x+h)^2+5)-(x^2+5)}{h}$$

Now expand the numerator to simplify the expression

$$\frac{(x^2+2xh+h^2+5)-(x^2+5)}{h} = \frac{2xh+h^2}{h} = 2x+h$$

## 4.6 Graphs of Functions

Determine whether $f$ is even, odd, or neither even nor odd.

$$f(x) = 8x^3 - 3x^2$$

## Solution

To determine whether the function is odd or even, recall that a function is even if $f(-x) = f(x)$ and a function is odd if $f(-x) = -f(x)$. Check to see if either of these equalities holds.

$$\begin{aligned} f(x) &= 8^3 - 3x^2 \\ f(-x) &= 8(-x)^3 - 3(-x)^2 = -8x^3 - 3x^2 \\ -f(x) &= -(8x^2 - 3x^2) = -8x^2 + 3x^2 \end{aligned}$$

We can see from here that $f(-x) \neq f(x)$ and $f(x) \neq -f(x)$, so the function is neither even nor odd.

### 4.7 Parabola

Express $f(x)$ in the form $a(x-h)^2+k$ and graph.

$$f(x) = -3x^2 - 6x - 5$$

## Solution

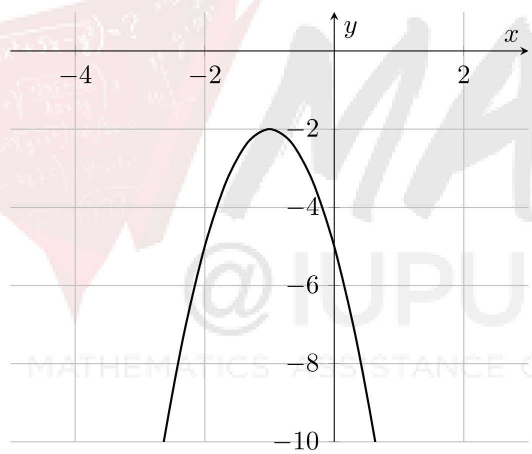
Begin by completing the square.

$$\begin{aligned} f(x) &= -3x^2 - 6x - 5 \\ &= -3(x^2+2x) - 5 \\ &= -3(x^2+2x+1) - 5 + 3 \\ f(x) &= -3(x+1)^2 - 2 \end{aligned}$$

To begin graphing, let $f(x) = y$ and notice the equation is the equation of a parabola.

$$\begin{aligned} y &= -3(x+1)^2 - 2 \\ (y+2) &= -3(x+1)^2 \end{aligned}$$

From here, it can be seen that the parabola opens downwards with vertex at $(-1, -2)$.



### 4.8 Composite Functions

For $f(x)$ and $g(x)$ below find

(a) $(f \circ g)$ and its domain.

(b) $(g \circ f)$ and its domain.

$$f(x) = \sqrt{3-x} \qquad\qquad g(x) = \sqrt{x^2-16}$$

## Solution

(a) Begin by plugging in the function $g(x)$ into $f(x)$.

$$(f \circ g)(x) = f(g(x)) = \sqrt{3-g(x)} = \sqrt{3-\sqrt{x^2-16}}$$

The domain of $(f \circ g)$ is everywhere that the inside of both square roots is non-negative, since we cannot take the square root of a negative number. So we have

$$\begin{aligned} 3-\sqrt{x^2-16} &\geq 0 \\ 3 &\geq \sqrt{x^2-16} \\ 9 &\geq x^2-16 \\ 25 &\geq x^2 \\ 5 &\geq |x| \\ -5 \leq &x \leq 5 \end{aligned} \qquad\qquad \begin{aligned} x^2-16 &\geq 0 \\ x^2 &\geq 16 \\ |x| &\geq 4 \\ x \geq 4 \text{ or } &x \leq -4 \end{aligned}$$

Combining these results in interval notation gives the domain to be $[-5,-4]\bigcup[4,5]$.

(b) Begin by plugging in the function $f(x)$ into $g(x)$.

$$(g \circ f)(x) = g(f(x)) = \sqrt{(f(x))^2-16} = \sqrt{(\sqrt{3-x})^2-16} = \sqrt{(3-x)-16} = \sqrt{-x-3}$$

To find the domain, set the inside of the square root to be greater than or equal to zero as above.

$$\begin{aligned} -x-13 &\geq 0 \\ -13 &\geq x \end{aligned}$$

So the domain, in interval notation, is $(-\infty,-13]$.

## 4.9 Polynomial Functions of Degree Greater than 2

Find all values of $x$ such that $f(x) > 0$ and all $x$ such that $f(x) < 0$, and sketch the graph of $f$.

$$f(x) = x^3 + 2x^2 - 4x - 8$$

## Solution

To find where the function is positive, begin by factoring the function as follows.

$$\begin{aligned} f(x) &= x^3 + 2x^2 - 4x - 8 \\ &= x^2(x+2) - 4(x+2) \\ &= (x^2 - 4)(x+2) \\ &= (x-2)(x+2)(x+2) \\ &= (x+2)^2(x-2) \end{aligned}$$

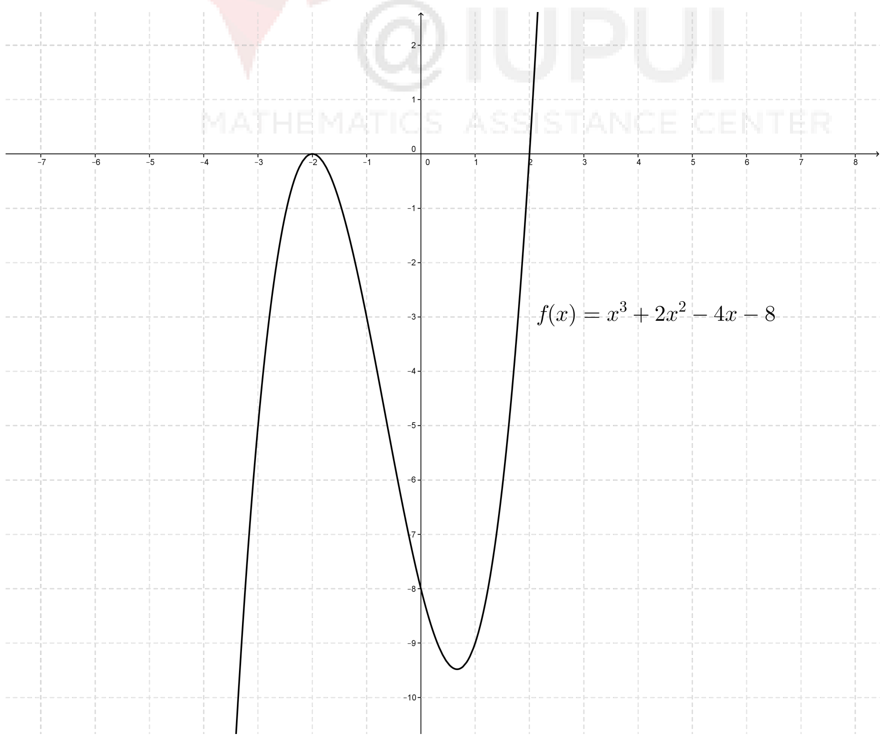
Since $(x+2)^2$ is a perfect square, it is always positive or zero. Because of this, the function is positive only when $(x-2) > 0$, or in other words, when $x > 2$. In interval notation, $f(x) > 0$ when $x$ is in the interval

$$(2, \infty)$$

To find where the function is negative, use the factorization above. It is easy to see that $f(x) < 0$ only when $(x-2) < 0$ and $x \neq -2$, since the term $(x+2)^2$ is always positive or equal to zero. Then in interval notation, $f(x) < 0$ when $x$ is in

$$(-\infty, -2) \cup (-2, 2)$$

To graph the function, use the fact that $f(x)$ is negative in $(-\infty, -2)$ and $(-2, 2)$ and $f(x)$ is positive in $(2, \infty)$.



# 5 Properties of Division

## 5.1 Long Division

Find the quotient $q(x)$ and remainder $r(x)$ if $f(x)$ is divided by $p(x)$.

$$f(x) = 3x^3 + 2x - 4 \qquad\qquad p(x) = 2x^2 + 1$$

### Solution

Use long division in the normal way to get

$$\begin{array}{r|l}
 & \tfrac{3}{2}x \\ \hline
2x^2+1 & 3x^3 + 2x - 4 \\
 & \underline{-\,3x^3 - \tfrac{3}{2}x} \\
 & \tfrac{1}{2}x
\end{array}$$

$$q(x) = \frac{3}{2}x \qquad\qquad r(x) = \frac{1}{2}x - 4$$

### 5.2 Synthetic Division

Use synthetic division to find the quotient and remainder if $f(x)$ is divided by $p(x)$

$$f(x) = 2x^3 - 3x^2 + 4x - 5 \qquad\qquad p(x) = x - 2$$

## Solution

Use the standard synthetic division procedure to obtain

$$\begin{array}{r|rrrr} & 2 & -3 & 4 & -5 \\ 2 & & 4 & 2 & 12 \\ \hline & 2 & 1 & 6 & 7 \end{array}$$

$$q(x) = 2x^2 + x + 6$$
$$r(x) = 7$$

# 6 Inverse Functions

## 6.1 Finding Inverse

Find the inverse function of $f$.

$$f(x) = \frac{3x+2}{2x-5}$$

## Solution

Begin by solving for $x$ in terms of $f(x)$.

$$\begin{aligned} f(x) &= \frac{3x+2}{2x-5} \\ (2x-5)f(x) &= 3x+2 \\ 2xf(x) - 5f(x) &= 3x+2 \\ 2xf(x) - 3x &= 5f(x)+2 \\ x(2f(x)-3) &= 5f(x)+2 \\ x &= \frac{5f(x)+2}{2f(x)-3} \end{aligned}$$

From here, we replace $x$ with $f^{-1}(x)$ and $f(x)$ with $x$ to get our answer.

$$f^{-1}(x) = \frac{5x+2}{2x-3}$$

### 6.2 Domain and Range of $f^{-1}$

Determine the domain and range of $f^{-1}$ for the given function.

$$f(x) = -\frac{4x+5}{3x-8}$$

## Solution

Since the domain of $f(x)$ is $\left(-\infty, \frac{8}{3}\right) \bigcup \left(\frac{8}{3}, \infty\right)$, we have that the range of $f^{-1}(x)$ is simply

$$\left(-\infty, \frac{8}{3}\right) \bigcup \left(\frac{8}{3}, \infty\right)$$

To find the domain of $f^{-1}(x)$, we first need to find $f^{-1}(x)$

$$\begin{aligned}
y &= -\frac{4x+5}{3x-8} \\
x &= -\frac{4y+5}{3y-8} \\
3yx - 8x &= -4y + 5 \\
y(3x+4) &= 8x + 5 \\
\frac{y\cancel{(3x+4)}}{\cancel{3x+4}} &= \frac{8x+5}{3x+4} \\
y &= \frac{8x+5}{3x+4}
\end{aligned}$$

The domain of $f^{-1}(x)$ is $\left(-\infty, -\frac{4}{3}\right) \bigcup \left(-\frac{4}{3}, \infty\right)$.

# 7 Exponential and Logarithmic Functions

## 7.1 Exponential Functions

Solve the equation.

(a) $3^{x+4} = 2^{1-3x}$

(b) $2^{2x-3} = 5^{x-2}$

## Solution

(a)

$$\begin{aligned}
3^{x+4} &= 2^{1-3x} \\
\log(3^{x+4}) &= \log(2^{1-3x}) \\
(x+4)\log(3) &= (1-3x)\log(2) \\
x\log(3) + 4\log(3) &= \log(2) - 3x\log(2) \\
x\log(3) + 3x\log(2) &= \log(2) - \log(3^4) \\
x\big[\log(3) + \log(2^3)\big] &= \log\left(\frac{2}{81}\right) \\
x &= \frac{\log\left(\frac{2}{81}\right)}{\log(3\times 8)} \\
x &= \frac{\log\left(\frac{2}{81}\right)}{\log(24)} \\
x &\approx -1.16
\end{aligned}$$

(b)

$$\begin{aligned}
2^{2x-3} &= 5^{x-2} \\
(2x-3)\log(2) &= (x-2)\log(5) \\
2x\log(2) - 3\log(2) &= x\log(5) - 2\log(5) \\
x\log(2^2) - x\log(5) &= \log(2^3) - \log(5^2) \\
x\big[\log(4) - \log(5)\big] &= \log\left(\frac{8}{25}\right) \\
x &= \frac{\log\left(\frac{8}{25}\right)}{\log\left(\frac{4}{5}\right)} \\
x &\approx 5.11
\end{aligned}$$

### 7.2 Compound Interest Formula

If \$1000 is invested at a rate of 12% per year compounded monthly, find the amount after:

(a) 1 month

(b) 6 months

(c) 1 year

(d) 20 years

## Solution

(a) Here, we need to use the following formula for calculating payments:

$$P = P_0\left(1+\frac{r}{n}\right)^{nt}$$

where $t$ is the number of periods that have passed (i.e. how many times interest has been compounded) and $n$ is the number of times interest is compounded annually. Thus:

$$P = 1000\left(1+\frac{0.12}{12}\right)^{\frac{1}{12}*12} = \$1010$$

(b) Similar to the previous part:

$$\begin{aligned} P &= P_0\left(1+\frac{r}{n}\right)^{nt} \\ &= 1000\left(1+\frac{0.12}{12}\right)^{6} \\ &= \$1061.52 \end{aligned}$$

(c) Here, we need to consider the number of months there are in a year rather than simply plugging in the number of years:

$$\begin{aligned} P &= 1000\left(1+\frac{0.12}{12}\right)^{12} \\ &= \$1126.83 \end{aligned}$$

(d) 20 years $= 20 \times 12 = 240$ months. Therefore:

$$\begin{aligned} P &= 1000\left(1+\frac{0.12}{12}\right)^{240} \\ &= \$10892.55 \end{aligned}$$

### 7.3 Continuously Compounded Interest Formula

If $P$ dollars is deposited in a savings account that pays interest at a rate of $r\%$ per year compounded continuously, find the balance after $t$ years.

$$P = 1000 \qquad r = 8\frac{1}{4} \qquad t = 5$$

## Solution

Here,we will use the formula for interested compounded continuously:

$$P = P_0 e^{rt}$$

Where $t$ is the number of years. First, we will write the rate as a fraction and then plug it into the formula:

$$r = 8\frac{1}{4} = \frac{8 \times 4 + 1}{4} = \frac{33}{4}\%$$
$$r = \frac{33}{4} \times \frac{1}{100} = \frac{33}{400}$$

Thus:

$$P = (1000)e^{(33/400)(5)}$$
$$= \$1,510.59$$

## 7.4 Natural Exponetial Function

Find the zeros of $f$.

$$f(x) = x^3(4e^{4x}) + 3x^2e^{4x}$$

## Solution

First, let us set up our equation:

$$x^3(4e^{4x}) + 3x^2e^{4x} = 0$$

now, since $e^{4x}$ is always positive and never zero, we can take it out as a common factor and divide by it:

$$\begin{aligned} e^{4x}(4x^3 + 3x^2) &= 0 \\ 4x^3 + 3x^2 &= \frac{0}{e^{4x}} \\ 4x^3 + 3x^2 &= 0 \end{aligned}$$

Now it is simple to factor the equation at hand and find the zeros:

$$x^2(4x + 3) = 0$$

$$\begin{array}{rcr} x^2 = 0 & \text{and} & 4x + 3 = 0 \\ x = 0 & \text{and} & 4x = -3 \\ x = 0 & \text{and} & x = -\dfrac{3}{4} \end{array}$$

The population $N(t)$ (in millions) of the United States $t$ years after 1980 may be approximated by the formula $N(t) = 227e^{0.007t}$.

(a) When will the population be twice what it was in 1980?

## Solution

We can see that the population in 1980 was 227 million (you can see this by simply plugging in $t = 0$). Now, we would like to find the time at which $N(t) = 454$:

$$\begin{aligned} 454 &= 227e^{0.007t} \\ 2 &= e^{0.007t} \\ \ln(2) &= 0.007t \\ t &= \frac{\ln(2)}{0.007} \approx 99 \end{aligned}$$

Thus, the population will be twice what it was in 1980 in the year $1980 + 99 =$ **2079**.

Use natural logarithms to solve for $x$ in terms of $y$.

$$y = \frac{e^x - e^{-x}}{2}$$

## Solution

We can begin by writing $e^{-x}$ as follows:

$$y = \frac{e^x - \frac{1}{e^x}}{2}$$

From here, we attempt to isolate the term $e^x$:

$$2y = e^x - \frac{1}{e^x}$$
$$2ye^x = e^{2x} - 1$$
$$0 = e^{2x} - 2ye^x - 1$$

Now using the quadratic formula:

$$e^x = \frac{2y \pm \sqrt{4y^2 + 4}}{2}$$
$$e^x = y \pm \sqrt{y^2 + 1}$$

Here, notice that $e^x$ is always positive, so we can use the positive answer. However, notice that the expression $y - \sqrt{y^2 + 1}$ is always negative since $y - \sqrt{y^2 + 1} < 0 \implies y^2 < y^2 + 1 \implies 0 < 1$ which is obviously true. Thus:

$$e^x = y + \sqrt{y^2 + 1}$$
$$x = \ln(y + \sqrt{y^2 + 1})$$

### 7.5 Properties of Logarithms

Solve for $t$ using logarithms with base $a$

$$A = Ba^{Ct} + D$$

## Solution

To do this, we simply isolate the term with $t$ in it as follows:

$$a^{Ct} = \frac{A - D}{B}$$

Now we can take the natural logarithm of base $a$ on both sides:

$$Ct = \log_a \left( \frac{A - D}{B} \right)$$
$$t = \frac{1}{C} \log_a \left( \frac{A - D}{B} \right)$$

Solve the equation

$$\ln(-4-x)+\ln 3=\ln(2-x)$$

## Solution

we will begin by simplifying the problem using the properties of logarithms:

$$\begin{aligned}\ln(-4-x)+\ln 3&=\ln(2-x)\\ \ln(3(-4-x))&=\ln(2-x)\end{aligned}$$

Now, we can exponentiate both sides of the equation to remove the logarithm:

$$\begin{aligned}e^{\ln(3(-4-x))}&=e^{\ln(2-x)}\\ 3(-4-x)&=2-x\\ -3x-12&=2-x\\ -14&=2x\\ x&=-7\end{aligned}$$

Since we exponentiated both sides of the equation, we must check our answers for extraneous solutions by plugging it back in:

$$\begin{aligned}\ln(-4-(-7))+\ln(3)&\stackrel{?}{=}\ln(2-(-7))\\ \ln(3)+\ln(3)&\stackrel{?}{=}\ln(9)\\ \ln(3\times 3)&=\ln(9)\end{aligned}$$

Thus, the result checks off and indeed, $x=-7$ is a solution.

Solve the equation

$$\log_3(x+3) + \log_3(x+5) = 1$$

## Solution

We will use the properties of logarithms to simplify the equation as follows:

$$\begin{aligned}\log_3(x+3) + \log_3(x+5) &= 1\\ \log_3((x+3)(x+5)) &= 1\end{aligned}$$

Here, we exponentiate both sides using a base of 3 to remove the logarithm:

$$\begin{aligned}3^{\log_3((x+3)(x+5))} &= 3^1\\ (x+3)(x+5) &= 3\\ x^2+8x+15 &= 3\\ x^2+8x+12 &= 0\end{aligned}$$

From here, we can either use the quadratic formula or factor the expression on the left hand side of the equation (we choose to factor here, but the quadratic formula will give the same solutions):

$$(x+2)(x+6) = 0$$

This means that either $x = -6$ or $x = -2$. We must check our answer since we exponentiated both sides of the equation in the process.

For $x = -2$ :

$$\begin{aligned}\log_3(-2+3) + \log_3(-2+5) &\stackrel{?}{=} 1\\ \log_3(1) + \log_3(3) &\stackrel{?}{=} 1\\ 0+1 &= 1\end{aligned}$$

So $x = -2$ is a valid solution.

For $x = -6$:

$$\begin{aligned}\log_3(-6+3) + \log_3(-6+5) &\stackrel{?}{=} 1\\ \log_3(-3) + \log_3(-1) &\stackrel{?}{=} 1\end{aligned}$$

Notice that the logarithm is not defined for negative values, thus $x = -6$ is not a valid solution. The only solution here is $x = -2$.

Find the exact solution, using common logarithms, and a two-decimal-place approximation, when appropriate.

$$\log(x-4) - \log(3x-10) = \log\left(\frac{1}{x}\right)$$

## Solution

Once again, we will use the same simplification step to rewrite the equation as follows:

$$\log(x-4) - \log(3x-10) = \log\left(\frac{1}{x}\right)$$
$$\log\left(\frac{x-4}{3x-10}\right) = \log\left(\frac{1}{x}\right)$$

Now, exponentiating both sides with a base 10:

$$10^{\log\left(\frac{x-4}{3x-10}\right)} = 10^{\log\left(\frac{1}{x}\right)}$$
$$\frac{x-4}{3x-10} = \frac{1}{x}$$

We can now cross multiply and proceed to solve the equation at hand:

$$x(x-4) = 3x-10$$
$$x^2 - 4x = 3x - 10$$
$$x^2 - 7x + 10 = 0$$

Once again, either the quadratic formula or factoring will work (we will use the quadratic formula this time around):

$$x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(1)(10)}}{2}$$
$$x = \frac{7 \pm 3}{2}$$

Hence, either $x = 5$ or $x = 2$. We must check both answers since we exponentiated both sides of the equation in the process.

For $x = 5$:

$$\log(5-4) - \log(3(5)-10) \stackrel{?}{=} \log\left(\frac{1}{5}\right)$$
$$\log(1) - \log(5) \stackrel{?}{=} -\log(5)$$
$$0 - \log(5) = -\log(5)$$

Thus, $x = 5$ is a valid solution.

For $x = 2$:

$$\log(2-4) - \log(3(2)-10) \stackrel{?}{=} \log\left(\frac{1}{2}\right)$$

This results in negative arguments in the logarithm function, which is not defined. Thus, $x = 2$ is not a valid solution, and the only solution is $x = 5$.

Find the exact solution, using common logarithms, and a two-decimal-place approximation, when appropriate.

$$4^x - 3(4^{-x}) = 8$$

## Solution

We will begin by rewriting $4^{-x}$ as follows:

$$\begin{aligned} 4^x - 3(4^{-x}) &= 8 \\ 4^x - \frac{3}{4^x} &= 8 \end{aligned}$$

Multiplying both sides by $4^x$ clears out the denominator:

$$\begin{aligned} (4^x)^2 - 3 &= 8(4^x) \\ (4^x)^2 - 8(4^x) - 3 &= 0 \end{aligned}$$

Notice that this is a quadratic equation, which means we can solve this by first making the substitution $u = 4^x$ and using the quadratic formula:

$$\begin{aligned} u^2 - 8u - 3 = 0 \\ u &= \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(-3)}}{2} \\ u &= \frac{8 \pm \sqrt{76}}{2} \\ u &== \frac{8 \pm \sqrt{4 \times 19}}{2} \\ u &= \frac{8 \pm 2\sqrt{19}}{2} \\ u &= 4 \pm \sqrt{19} \end{aligned}$$

Now we must substitute back the expression $u = 4^x$ to attain the solutions in terms of x:

$$\begin{aligned} 4^x &= 4 + \sqrt{19} \qquad \text{or} \qquad 4^x = 4 - \sqrt{19} \\ x &= \log_4(4 + \sqrt{19}) \qquad \text{or} \qquad x = \log_4(4 - \sqrt{19}) \end{aligned}$$

Notice that $4 - \sqrt{19}$ is a negative value, and thus we cannot apply the logarithm to it. thus, the only solution is:

$$x = \log_4(4 + \sqrt{19}) \approx 1.53$$

# 8 Conics

## 8.1 Parabolas

Find the vertex, focus, and directrix of the parabola. Sketch its graph, showing the focus and the directrix.

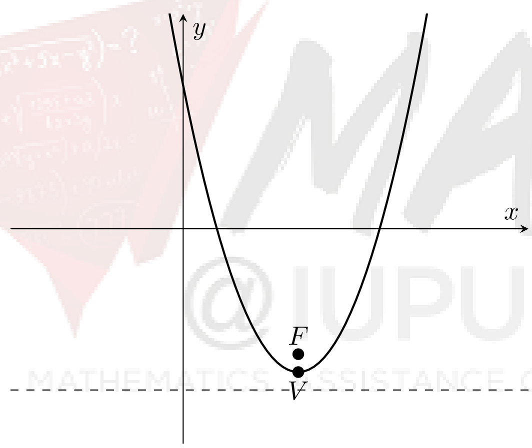
### Solution

Begin the problem by completing the square.

$$\begin{aligned} y &= x^2 - 4x + 2 \\ y - 2 &= x^2 - 4x \\ y - 2 + 4 &= x^2 - 4x + 4 \\ y + 2 &= (x-2)^2 \end{aligned}$$

Now our equation is of the form $4p(y-k) = (x-h)^2$

So the vertex of our parabola $V(h,k)$ is $(2,-2)$. Now $4p = 1$ so $p = \dfrac{1}{4}$. Our parabola is vertical since $x$ is the squared term. Our focus $F(h, k+p)$ is $(2, -2+\dfrac{1}{4}) = (2, -\dfrac{7}{4})$. Lastly, our directrix is $y = k-p = -2-\dfrac{1}{4} = -\dfrac{9}{4}$.



Find an equation of the parabola that satisfies the given conditions.

$$\text{Focus } F(6,4), \text{ directrix } y=-2$$

## Solution

If our directrix is $y=-2$, then our general equation for the parabola is:

$$4p(y-k)=(x-h)^2$$

And $y=-2=k-p$. Furthermore, the focus is $F(h,k+p)=(6,4)$ so $h=6$ and $k+p=4$. If $k+p=4$ and $-2=k-p$, we can find that solve for $k$ and $p$ by adding the two equations:

$$\begin{aligned}&k+p=4\\&k-p=-2\\&\implies 2k=2 \implies k=1\end{aligned}$$

Plugging the value of $k$ back in, we see that $p=3$. Putting all of the pieces together, the equation of the parabola is:

$$12(y-1)=(x-6)^2$$

Find an equation of the parabola that satisfies the given conditions.

Vertex $V(-3, 5)$, axis parallel to the $x$-axis, and passing through the point (5,9)

## Solution

If the axis is parallel to the $x$-axis, then our parabola is horizontal and is of the form:

$$(y-k)^2 = 4p(x-h)$$

The vertex $V(h, k) = (-3, 5)$.

Let's plug in the values that we already know.

$$(y-5)^2 = 4p(x+3)$$

We can plug in the additional point we were given so that we can solve for $p$.

$$(9-5)^2 = 4p(5+3)$$

Solving this equation, we can see that

$$p = \frac{1}{2}$$

So our final equation is

$$(y-5)^2 = 2(x+3)$$

### 8.2 Ellipses

Find the vertices and foci of the ellipse. Sketch its graph, showing the foci.

$$\frac{(x-3)^2}{16}+\frac{(y+4)^2}{9}=1$$

## Solution

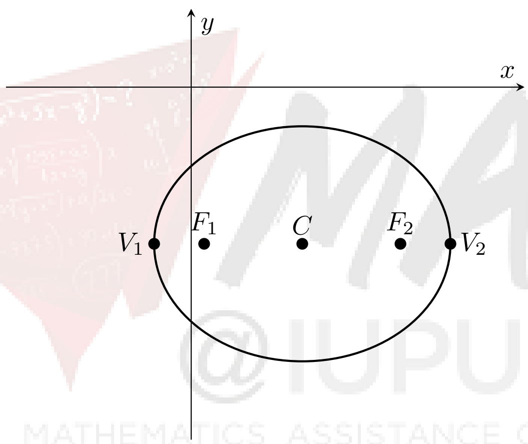
This ellipse is in the form:

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

Now we can see that the center $C(h,k)=(3,-4)$ and the major axis runs horizontal away from the center by $a=4$ in both directions. This gives our vertices as:

$$(3\pm 4,-4)$$

We know our foci are $(h\pm c,k)$ where $c^2=a^2-b^2=16-9=7$. Thus, our foci are $(3\pm\sqrt{7},-4)$. See the sketch below.



Find the vertices and foci of the ellipse. Sketch its graph, showing the foci.

$$4x^2 + 9y^2 - 32x - 36y + 64 = 0$$

## Solution

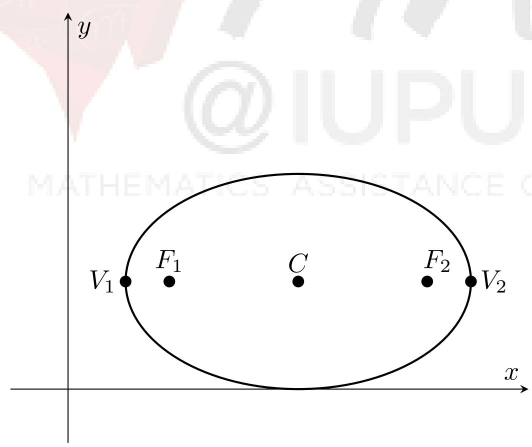
Begin the problem by completing the square for $x$ and $y$

$$\begin{aligned}
4x^2 + 9y^2 - 32x - 36y + 64 &= 0 \\
4x^2 - 32x + 9y^2 - 36y &= -64 \\
4(x^2 - 8x) + 9(y^2 - 4y) &= -64 \\
4(x^2 - 8x) + 64 + 9(y^2 - 4y) + 36 &= -64 + 64 + 36 \\
4(x^2 - 8x + 16) + 9(y^2 - 4y + 4) &= 36 \\
4(x-4)^2 + 9(y-2)^2 &= 36 \\
\frac{4(x-4)^2}{36} + \frac{9(y-2)^2}{36} &= 1 \\
\frac{(x-4)^2}{9} + \frac{(y-2)^2}{4} &= 1
\end{aligned}$$

By completing the squares, our equation is in the desired form of

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

Our center is $C(h,k) = (4,2)$ and our vertices are $V(h \pm a, k) = (4 \pm 3, 2)$. The foci are $(h \pm c, k)$ where $c^2 = a^2 - b^2 = 9 - 4 = \sqrt{5}$. Thus, our foci are $(4 \pm \sqrt{5}, 2)$.



Find an equation of the ellipse that has its center at the origin and satisfies the given conditions.

$$\text{Vertices } V(\pm 8, 0), \text{ foci } F(\pm 5, 0)$$

## Solution

Notice that the $\pm$ being in the $x$-coordinates tell us that the major axis is horizontal. Then our general equation is:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

With foci $(h \pm c, k)$ and vertices $V(h \pm a, k)$. Studying our vertices and foci, we can see that $h = 0$ and $k = 0$. Furthermore, $a = 8$ and $c = 5$. We can use $a$ and $c$ to find $b$. If $a^2 - b^2 = c^2$, we see that $b = \sqrt{39}$. Finally, plugging everything in, our equation is :

$$\frac{x^2}{64} + \frac{y^2}{39} = 1$$

## 8.3 Hyperbolas

Find the vertices, the foci, and the equation of the asymptotes of the hyperbola. Sketch its graph, showing the asymptotes and the foci.

$$4y^2 - x^2 + 40y - 4x + 60 = 0$$

## Solution

$$\begin{aligned}
4y^2 - x^2 + 40y - 4x + 60 &= 0 \\
4y^2 + 40y - x^2 - 4x &= -60 \\
4(y^2 + 10y) - 1(x^2 + 4x) &= -60 \\
4(y^2 + 10y) + 100 - 1(x^2 + 4x) - 4 &= -60 + 100 - 4 \\
4(y^2 + 10y + 25) - 1(x^2 + 4x + 4) &= 36 \\
4(y+5)^2 - (x+2)^2 &= 36 \\
\frac{4(y+5)^2}{36} - \frac{(x+2)^2}{36} &= 1 \\
\frac{(y+5)^2}{9} - \frac{(x+2)^2}{36} &= 1
\end{aligned}$$

Our hyperbola is in the form:

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

Our hyperbola is centered at $(h, k) = (-2, -5)$ and since $y$ is in the positive term, the hyperbola is vertical. Additionally, we can see that $a = 3$ and $b = 6$. This means that our major axis runs vertically and our vertices are

$$(-2, -5 \pm 3)$$

The foci are found using $(h, k \pm c)$ where $c^2 = a^2 + b^2 = 9 + 36 = 45$ so they become:

$$(-2, -5 \pm 3\sqrt{5})$$

Now, let us find the asymptotes. Using the auxillary rectangle, we know that the asymptotes go through the center and the corners of the rectangle. The corners are $\pm b$ away from the center horizontally and $\pm a$ away vertically. Then the slopes are $\pm\frac{a}{b}$. Using point-slope form, we'll get:

$$\begin{aligned}
y + 5 &= \frac{3}{6}(x+2) \\
y + 5 &= \frac{x}{2} + 1 \\
y &= \frac{x}{2} - 4
\end{aligned}
\qquad\qquad
\begin{aligned}
y + 5 &= -\frac{3}{6}(x+2) \\
y + 5 &= -\frac{x}{2} - 1 \\
y &= -\frac{x}{2} - 6
\end{aligned}$$

Find an equation of the hyperbola that has its center at the origin and satisfies the given conditions.

$$\text{Vertices } V(\pm 4, 0), \text{ passing through } (8, 2)$$

## Solution

Our parabola is horizontal since its vertices are $\pm$ on the $x$-axis. This tell us that the general equation is $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ and $V(h \pm a, k) = (0 \pm 4, 0)$ which tell us that $(h, k) = (0, 0)$ and $a = 4$. Now our equation becomes $\frac{x^2}{16} - \frac{y^2}{b^2} = 1$. To find $b$, we can use the point that we're given.

$$\begin{aligned} \frac{(8)^2}{16} - \frac{(2)^2}{b^2} &= 1 \\ \frac{64}{16} - \frac{4}{b^2} &= 1 \\ 4 - \frac{4}{b^2} &= 1 \\ -\frac{4}{b^2} &= -3 \\ \frac{4}{b^2} &= 3 \\ \frac{4}{3} &= b^2 \end{aligned}$$

Finally, our equation comes $\frac{x^2}{16} - \frac{y^2}{\frac{4}{3}} = 1$ which is equivalent to $\frac{x^2}{16} - \frac{3y^2}{4} = 1$

# 9 Systems of Equations

## 9.1 Elimination and Substitution

Use the method of substitution to solve the system.

$$\begin{cases} x^2 + y^2 = 16 \\ 2y - x = 4 \end{cases}$$

### Solution

Let's solve the second equation for $x$ giving us $x = 2y - 4$. Now, we can plug it in for $x$ in the first equation and solve for $y$.

$$\begin{aligned} (2y-4)^2 + y^2 &= 16 \\ 4y^2 - 16y + 16 + y^2 &= 16 \\ 5y^2 - 16y &= 0 \\ y(5y - 16) &= 0 \\ y &= \frac{16}{5}, 0 \end{aligned}$$

Now, we can plug these values into either equation (the second will be easier). **If you plug them into an equation with any terms squared or higher power, you may get extraneous answers. This is fine as long as you double-check each answer gotten this way at the end.**

$$\begin{aligned} 2(\frac{16}{5}) - x &= 4 \\ \frac{32}{5} - x &= 4 \\ -x &= -\frac{12}{5} \\ x &= \frac{12}{5} \end{aligned}$$

Which makes one solutions $\left(\frac{12}{5}, \frac{16}{5}\right)$. Next:

$$\begin{aligned} 2(0) - x &= 4 \\ -x &= 4 \\ x &= -4 \end{aligned}$$

So our second solution is $(-4, 0)$.

Use the method of substitution to solve the system.

$$\begin{cases} y^2 - 4x^2 = 4 \\ 9y^2 + 16x^2 = 140 \end{cases}$$

## 9.2 Solution

Let's solve the first equation for $y^2$. Note that I won't solve $y$ as we'll have to square it when we substitute it anyhow. $y = 4x^2 + 4$ and we can substitute this into the second equation.

$$\begin{aligned} 9(4x^2 + 4) + 16x^2 &= 140 \\ 36x^2 + 36 + 16x^2 &= 140 \\ 52x^2 &= 104 \\ x^2 &= 2 \\ x &= \pm\sqrt{2} \end{aligned}$$

**The $\pm$ is crucial!** Now let's plug both of these values into either equation.

$$\begin{aligned} x &= \sqrt{2} \\ y^2 - 4(\sqrt{2})^2 &= 4 \\ y^2 - 8 &= 4 \\ y^2 &= 12 \\ y &= \pm 2\sqrt{3} \end{aligned}$$

This gives us two solutions $(\sqrt{2}, 2\sqrt{3})$ and $(\sqrt{2}, -2\sqrt{3})$. Now let $x = -\sqrt{2}$:

$$\begin{aligned} y^2 - 4(-\sqrt{2})^2 &= 4 \\ y^2 - 8 &= 4 \\ y^2 &= 12 \\ y &= \pm 2\sqrt{3} \end{aligned}$$

which gives us another two solutions $(-\sqrt{2}, 2\sqrt{3})$ and $(-\sqrt{2}, -2\sqrt{3})$.
Plugging these into either equation can confirm that all four are solutions. **Do not list these solutions as $(\pm 2, \pm 2\sqrt{3})$. This is technically different.** It's best to list out each solution explicitly.

Use the method of substitution to solve the system.

$$\begin{cases} x = y^2 - 4y + 5 \\ x - y = 1 \end{cases}$$

Solving for $x$ in the second equation gives $x = y + 1$. Plug this into the first equation and solve for $y$.

$$\begin{aligned} (y+1) &= y^2 - 4y + 5 \\ 0 &= y^2 - 5y + 4 \\ (y-1)(y-4) &= 0 \\ y &= 1, 4. \end{aligned}$$

We can plug these into the second equation to find the $x$ values:

$$y = 1 \implies x - (1) = 1 \implies x = 2$$
Thus the first solution is $(2, 1)$

Now for $y = 4$:

$$y = 4 \implies x - (4) = 1 \implies x = 5$$
Thus the second solution is $(5, 4)$

## 9.3 Applied Problems

The price of admission to a high school play was \$3.00 for students and \$4.50 for nonstudents. If 450 tickets were sold for a total of \$1555.50, how many of each kind were purchased?

## Solution

Let $x$ be the number of student tickets sold and $y$ the number of nonstudent tickets sold. We know that $x + y = 450$. Considering each of their costs, we also know that $3(x) + 4.5(y) = 1555.50$. This gives us our system of equations.

$$\begin{cases} x + y = 450 \\ 3x + 4.5y = 1555.50 \end{cases}$$

Solving for $x$ in the first equation, $x = 450 - y$. Now plugging this into the second equation:

$$\begin{aligned} &3(450 - y) + 4.5y = 1555.50 \\ &1350 - 3y + 4.5y = 1555.50 \\ &1.5y = 205.5 \\ &y = 137 \end{aligned}$$

and plugging this back into the first equation, we find that $x = 313$. Thus, 313 student tickets and 137 nonstudent tickets were sold.

A small furniture company manufactures sofas and recliners. Each sofa requires 8 hours of labor and \$60 in materials, while a recliner can be built for \$35 in 6 hours. The company has 340 hours of labor available each week and can afford to buy \$2250 worth of materials. How many recliners and sofas can be produced if all labor hours and all materials must be used?

## Solution

Let $x$ be the number of sofas built and $y$ the number of recliners built each week. Knowing how many hours each takes to be built, we can see that given a 340 hour limit each week, $8x + 6y = 340$. Similarly, we know the cost of materials for each and the total money available giving the second equation $60x+35y = 2250$.

$$\begin{cases} 8x + 6y = 340 \\ 60x + 35y = 2250 \end{cases}$$

Either method of solving this system would work. We're going to use elimination here. Particularly, we are going to eliminate $y$ by multiplying the first equation by $-35$ and the second by 6.

$$\begin{cases} -35(8x + 6y) = -35(340) \\ 6(60x + 35y) = 6(2250) \end{cases}$$

$$\begin{cases} -280x - 210y = -11900 \\ 360x + 210y = 13500 \end{cases}$$

Adding these together, we get $80x = 1600$ and $x = 20$. Plugging this into the first equation:

$$\begin{aligned} &8(20) + 6y = 340 \\ &160 + 6y = 340 \\ &6y = 180 \\ &y = 30. \end{aligned}$$

Therefore, 20 sofas and 30 recliners can be produced.

# 10 Angles and Speeds

## 10.1 Arcs and Sectors

Given $s = 7$ cm and $r = 4$ cm, answer the following.

(a) Find the radian and degree measures of the central angle $\theta$ subtended by the given arc of length $s$ on a circle of radius $r$.

### Solution

The equation we need here is $s = r\theta$, where $s$ is the subtended arc, $r$ is the radius, and $\theta$ is the central angle **IN RADIANS**.
Plugging in the info we were given, $7 = 4\theta$ so $\theta = \frac{7}{4}$ (in radians).

(b) Find the area of the sector determined by $\theta$.

### Solution

Now the equation we are looking for is $A = \frac{1}{2}r^2\theta$ where $A$ is the area of the sector, $r$ is the radius, and $\theta$ is the central angle **in radians**.
$A = \frac{1}{2}(4 \text{ cm})^2(\frac{7}{4}) = 14 \text{ cm}^2$

## 10.2 Angular and Linear Speed

Given a radius of 5 in. and 40 rpm, answer the following.

(a) Find the angular speed (in radians per minute).

### Solution

The relationship we need is $\frac{s}{t} = r \cdot \frac{\theta}{t}$ where $\frac{s}{t}$ is the linear speed and $\frac{\theta}{t}$ is the angular speed. Note the only difference is a factor or $r$. Also, note that $\theta$ must be in radians. The first question asks us for the angular velocity in radians/min but gives us the angular velocity in rpm. All we need to do is convert knowing that 1 rev. $= 2\pi$ so $\frac{40 \text{ rev.}}{\text{min.}} \cdot \frac{2\pi}{1 \text{ rev.}} = 80\pi \frac{\text{rad.}}{\text{min.}}$.

(b) Find the linear speed of a point on the circumference (in ft/min).

### Solution

Now, given the angular velocity in the correct units, all we need to do is multiply by our radius and adjust the units to give us the linear velocity. $80\pi \cdot (5 \text{ in}) \cdot \frac{1 \text{ ft.}}{12 \text{ in.}} = \frac{100\pi}{3} \frac{\text{ft.}}{\text{min.}}$.

# 11 Values of Trigonometric Functions

## 11.1 Exact Values

Find the exact value.

(a) $\csc\left(\frac{3\pi}{4}\right)$

### Solution

First, we know that $\csc(x) = \frac{1}{\sin(x)}$. We also know that $\csc(\theta) = \pm\csc(\theta_{ref})$. So, first we look at the reference angle for $\theta = \frac{3\pi}{4}$. This angle is in the second quadrant and the remaining angle to the $x$ axis gives us a $\theta_{ref} = \frac{\pi}{4}$. So now $\csc(\frac{3\pi}{4}) = \pm\csc(\frac{\pi}{4})$. We decide the $\pm$ by knowing if the trigonometric function should be positive or negative with the original angle. $\frac{3\pi}{4}$ is in the second quadrant and $\csc(x)$ is positive when $\sin(x)$ is positive so $\csc(\frac{3\pi}{4}) = +\csc(\frac{\pi}{4}) = \frac{1}{\sin(\frac{\pi}{4})} = \frac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2}$.

(b) $\csc\left(-\frac{2\pi}{3}\right)$

### Solution

This problem could be done exactly the way part $a$) was done from the beginning, but we'll run through a different method. This requires knowing even and odd functions. $\csc(x)$ is an odd function meaning that $\csc(-x) = -\csc(x)$. So $\csc(-\frac{2\pi}{3}) = -\csc(\frac{2\pi}{3}$. Now using the same technique in part $a$) and knowing that $\frac{2\pi}{3}$ is in the second quadrant, we see that $-\csc(\frac{2\pi}{3}) = -(+\csc(\frac{\pi}{3})) = \frac{-1}{\sin(\frac{\pi}{3})} = \frac{-1}{\frac{\sqrt{3}}{2}} = \frac{-2}{\sqrt{3}}$ and rationalized $\frac{-2\sqrt{3}}{3}$.

## 11.2 Approximate Values

Approximate, to the nearest 0.01 radian, all angles $\theta$ in the interval $[0, 2\pi)$ that satisfy the equation.

(a) $\sin(\theta) = 0.4195$

### Solution

$\theta = \sin^{-}(0.4195) \approx 0.43 = \theta_1$ which is in the first quadrant. This is one of two angles on our interval. The other angle will be in the second quadrant (the other quadrant where *sin* is positive). The second angle will be $\theta_2 = \pi - 0.43 \approx 2.71$.

(b) $\tan(\theta) = -3.2504$

### Solution

$\theta = \tan^{-1}(-3.2504) \approx -1.27$. We know that $tan^{-1}$ has a range of $(-\frac{\pi}{2}, \frac{\pi}{2})$ so this angle must be in the fourth quadrant. This is angle is also not in our interval so we'll need to use its reference angle to find the correct solutions. Since $-1.27$ is in the fourth quadrant, $\theta_{ref} = 1.27$ and we can use this reference angle to find our solutions. We know *tan* is negative in the second and fourth quadrants. These solutions are $\theta_1 = \pi - 1.27 \approx 1.87$ and $\theta_2 = 2\pi - 1.27 \approx 5.01$.

(c) $\sec(\theta) = 1.7452$

### Solution

$\sec(\theta) = \frac{1}{\cos(\theta)} = 1.7452$. Then $\cos(\theta) = \frac{1}{1.7452} \approx 0.57$. So $\theta = \cos^{-1}(0.57) = 0.96 = \theta_1$. The first angle is in the first quadrant and the other quadrant where cos is positive is in the fourth quadrant so $\theta_2 = 2\pi - 0.96 \approx 5.32$.

## 11.3 Fundamental Identities

Use the fundamental identities to write the first expression in terms of the second, for any acute angle $\theta$.

$$\cot(\theta),\ \sin(\theta)$$

## Solution

$\cot(\theta) = \dfrac{\cos(\theta)}{\sin(\theta)}$. Since $\sin^2(\theta) + \cos^2(\theta) = 1$, we can solve for $\cos(\theta)$ and substitute. We can see that $\cos(\theta) = \pm\sqrt{1 - \sin^2(\theta)}$, but $\theta$ is acute so it's in the first quadrant and $\cos(\theta)$ must be positive. Therefore, $\cos(\theta) = \sqrt{1 - \sin^2(\theta)}$. Now, we substitute, $\cot(\theta) = \dfrac{\sqrt{1 - \sin^2(\theta)}}{\sin(\theta)}$.

# 12 Trigonometric Equations and Identities

## 12.1 Graphs of Trigonometric Functions

Find the amplitude, period, and phase shift and sketching the graph.

$$y = -2\sin(3x - \pi)$$

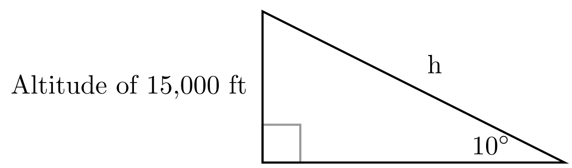
### Solution

The standard form for these equations is $y = a\sin(bx + c) + d$, where $|a|$ is amplitude, $\frac{2\pi}{|b|}$ is the period, $\frac{-c}{b}$ is the phase shift, and $d$ is the vertical shift. So, we can see that the amplitude is $|-2| = 2$, the period is $\frac{2\pi}{|3|} = \frac{2\pi}{3}$, and the phase shift is $\frac{-(-\pi)}{3} = \frac{\pi}{3}$. See below for the sketch.

## 12.2 Applied Problems in Trig.

An airplane takes off at a 10° angle and travels at a rate of 250 ft/sec. Approximately how long does it takes the airplane to reach an altitude of 15,000 feet?

## Solution




We can see that there is a trigonometric relationship in this right triangle. $\sin(10°) = \dfrac{15,000}{h}$. So $h = \dfrac{15000}{\sin(10°)} \approx 86381.56ft$, but we're not done because the question asked about how long would take to reach the required altitude. Knowing the rate the plane was travelling at, we can calculate this time since:

$$\text{distance} = \text{rate} \cdot \text{time}$$

$$\text{time} = \frac{\text{distance}}{\text{rate}}. \text{ So } t = \frac{86381.56\text{ft.}}{250\frac{\text{ft.}}{\text{sec.}}} \approx 346 \text{ seconds.}$$

## 12.3 Verifying Identities

Verify the identity.

$$\sec(\theta) - \cos(\theta) = \tan(\theta)\sin(\theta)$$

## Solution

First, it's important to note that there are **many ways** to prove these identities. Techniques that should be considered are converting everything to sines and cosines, substituting for squared terms, and making common denominators. For the above identity, let's begin by converting both sides to sines and cosines.

$$\frac{1}{\cos(\theta)} - \cos(\theta) = \frac{\sin(\theta)}{\cos(\theta)}\sin(\theta)$$

Now we can make common denominators on the left side.

$$\begin{aligned} \frac{1-\cos^2(\theta)}{\cos(\theta)} &= \tan(\theta)\sin(\theta) \\ \frac{\sin^2(\theta)}{\cos(\theta)} &= \tan(\theta)\sin(\theta) \\ \frac{\sin(\theta)}{\cos(\theta)}\sin(\theta) &= \tan(\theta)\sin(\theta) \\ \tan(\theta)\sin(\theta) &= \tan(\theta)\sin(\theta) \end{aligned}$$

Verify the identity.

$$\frac{1}{1-\cos(\gamma)}+\frac{1}{1+\cos(\gamma)}=2\csc^2(\gamma)$$

## Solution

We can begin by making common denominators on the left side.

$$\begin{aligned}
\frac{1+\cos(\gamma)}{(1-\cos(\gamma)(1+\cos(\gamma))}+\frac{1-\cos(\gamma)}{(1+\cos(\theta)(1-\cos(\gamma))} &= 2\csc^2(\gamma)\\
\frac{1+\cos(\gamma)+1-\cos(\gamma)}{1-\cos^2(\gamma)} &= 2\csc^2(\gamma)\\
\text{but we know that } 1-\cos^2(\gamma) &= \sin^2(\gamma)\\
\frac{2}{\sin^2(\gamma)} &= 2\csc^2(\gamma)
\end{aligned}$$

Verify the identity.

$$\tan^4(k) - \sec^4(k) = 1 - 2\sec^2(k)$$

## Solution

Fourth powers are hard to work with so we consider how we can reduce them and notice that the left side is a difference of squares so we now have:

$$\begin{aligned}
&(\tan^2(k) - \sec^2(k)) \cdot (\tan^2(k) + \sec^2(k)) = 1 - 2\sec^2(k) \\
&\text{Since } \tan^2(k) + 1 = \sec^2(k) \implies \tan^2(k) - \sec^2(k) = -1 \\
&(-1) \cdot (\tan^2(k) + \sec^2(k)) = 1 - 2\sec^2(k)
\end{aligned}$$

but the right-hand side only has secant in it, so we can subtitute in for the tangent on the left side.

$$\begin{aligned}
&(-1)((\sec^2(k) - 1) + \sec^2(k)) = 1 - 2\sec^2(k) \\
&(-1)(2\sec^2(k) - 1) = 1 - 2\sec^2(k)
\end{aligned}$$

### 12.4 Finding Solutions of Trig. Equations

Find all solutions to the equation.

$$\sin\left(2x - \frac{\pi}{3}\right) = \frac{1}{2}$$

## Solution

First, let us consider $\sin(\theta) = \frac{1}{2}$ where $\theta = 2x - \frac{\pi}{3}$. We know that $\theta_{ref}$ is $\frac{\pi}{6}$ from the special 30-60-90 triangle, and there will be two solutions, $\theta_1$ and $\theta_2$, in the two quadrants where sine is positive $+2\pi n$ . In quadrant 1, $\theta_1 = \frac{\pi}{6}$ and in quadrant 2, $\theta_2 = \frac{5\pi}{6}$. Our original question asked about $x$ so we can solve for $x$ now.
So, if $\theta_1 = \frac{\pi}{6} + 2\pi n$ then $2x_1 - \frac{\pi}{3} = \frac{\pi}{6} + 2\pi n$ and we can solve for $x_1$.

$$\begin{aligned}
2x_1 &= \frac{\pi}{6} + \frac{\pi}{3} + 2\pi n \\
2x_1 &= \frac{\pi}{2} + 2\pi n \\
x_1 &= \frac{\pi}{4} + \pi n.
\end{aligned}$$

Now we can do the same thing with $\theta_2$ to find $x_2$.

$$\begin{aligned}
2x_2 - \frac{\pi}{3} &= \frac{5\pi}{6} + 2\pi n \\
2x_2 &= \frac{5\pi}{6} + \frac{\pi}{3} + 2\pi n \\
2x_2 &= \frac{7\pi}{6} + 2\pi n \\
x_2 &= \frac{7\pi}{12} + \pi n
\end{aligned}$$

Find the solutions that are in the interval $[0, 2\pi)$.

$$2\tan(t) - \sec^2(t) = 0$$

## Solution

To solve this equation, we first need all of the trigonometric functions to be the same. We can make a substitution for $\sec^2(t)$ since we know that:

$$\sec^2(t) = \tan^2(t) + 1.$$

$$\begin{aligned} 2\tan(t) - (\tan^2(t) + 1) &= 0 \\ -\tan^2(t) + 2\tan(t) - 1 &= 0 \\ \tan^2(t) - 2\tan(t) + 1 &= 0 \quad \text{and now using the quadratic equation:} \\ \tan(t) = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(1)}}{2(1)} &= \frac{2 \pm \sqrt{0}}{2} = 1 \end{aligned}$$

We know that if $\tan(t) = 1$, $t$ must be in the first or third quadrant. Additionally, this is a special ratio from a 45-45-90 triangle. Then, we can conclude that **all** solutions would be $t = \frac{\pi}{4} + \pi n$, but our solutions must be in the interval $[0, 2\pi)$, so we can see that $t = \left\{\frac{\pi}{4}, \frac{5\pi}{4}\right\}$.

Approximate, to the nearest 10', the solutions in the interval $[0°, 360°)$.

$$\sin^2(t) - 4\sin(t) + 1 = 0$$

## Solution

We begin with the quadratic equation.

$\sin(t) = \dfrac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(1)}}{2(1)} = \dfrac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$ Since these do not relate to any special triangles, we can use inverse trig functions to find the approximate solutions.

Since $\sin(t) \leq 1$ for all $t$ and $1 < 2 + \sqrt{3}$, there are no solutions for this value.

Now, for the other value:

$$\sin(t) = 2 - \sqrt{3}$$
$$t = \sin^{-1}(2 - \sqrt{3}) \approx 15.54° \approx 15°30'$$

Since we are looking for all solutions on $[0°, 360°)$ and we know $\sin^{-1}$ will only give us one of the two, we need to consider the other solution. Since $2 - \sqrt{3} > 0$ and our first solution was in the first quadrant, we need the other quadrant where sine is positive, the second quadrant. So our second solution is $180° - 15°30' = 164°30'$. $t = \{15°30', 164°30'\}$.

# 13 Inverse Trigonometric Functions And Multiple Angle Formulas

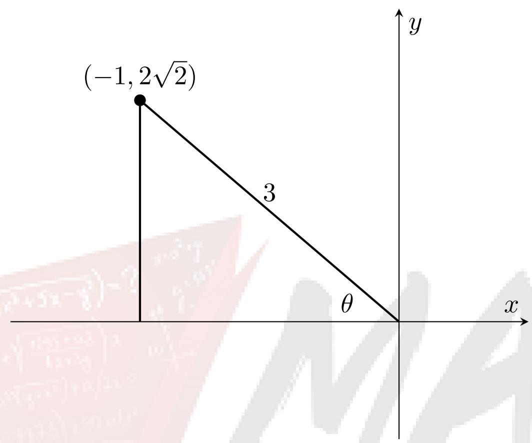
## 13.1 Double Angle Formulas

Find the exact values of $\sin(2\theta)$, $\cos(2\theta)$, and $\tan(2\theta)$ given the information below.

$$\sec(\theta) = -3, \quad 90^\circ < \theta < 180^\circ$$

### Solution

First, we can see that $\cos(\theta) = -\dfrac{1}{3}$ gives us:




We know that $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ and using the triangle and the fact that $\theta$ is in the second quadrant:

$$\sin(2\theta) = 2\left(\frac{2\sqrt{2}}{3}\right)\left(-\frac{1}{3}\right) = \frac{-4\sqrt{2}}{9}$$

$$\cos(2\theta) = 2\cos^2(\theta) - 1 = 2\left(-\frac{1}{3}\right)^2 - 1 = \frac{2}{9} - 1 = \frac{-7}{9}$$

$$\tan(2\theta) = \frac{2\tan(\theta)}{1 - \tan^2(\theta)} = \frac{2(-2\sqrt{2})}{1 - (-2\sqrt{2})^2} = \frac{-4\sqrt{2}}{1-8} = \frac{-4\sqrt{2}}{-7} = \frac{4\sqrt{2}}{7}$$

Use inverse trigonometric functions to find the solutions of the equation that are on $[0, 2\pi)$, and approximate solutions to four decimal places.

$$\cos^2(x) + 2\cos(x) - 1 = 0$$

## Solution

Using the quadratic formula:

$$\begin{aligned} \cos(x) &= \frac{-(2) \pm \sqrt{(2)^2 - 4(1)(-1)}}{2(1)} \\ &= \frac{-2 \pm \sqrt{8}}{2} = \frac{-2 \pm 2\sqrt{2}}{2} \\ \cos(x) &= -1 \pm \sqrt{2}. \end{aligned}$$

Since these do not relate to any special triangles, we can use inverse trig functions to find the approximate solutions.

$$\begin{aligned} x &= \cos^{-1}(-1-\sqrt{2}) \quad \text{does not exist since } -1-\sqrt{2} < -1, \text{ Thus:} \\ x_1 &= \cos^{-1}(-1+\sqrt{2}) \approx 1.1427 \end{aligned}$$

This is in the first quadrant since $-1 + \sqrt{2} > 0$. Note that $x_1 = x_{\text{ref}}$ since we are in the first quadrant. The other solution we are looking for has the same reference angle and is in the other quadrant where cosine is positive, the fourth quadrant. So, the other solution is

$$x_2 = 2\pi - x_{\text{ref}} = 2\pi - 1.1437 \approx 5.1395$$

Find the solutions that are in the interval $[0, 2\pi)$.

$$\sin(2t) + \sin(t) = 0$$

## Solution

Our first step is to rewrite $\sin(2t)$ so that all of the arguments (the inside of the trig functions) are the same. Using a double angle formula, we get:

$$\begin{aligned} 2\sin(t)\cos(t) + \sin(t) &= 0 \\ \sin(t)[2\cos(t) + 1] &= 0 \\ \sin(t) = 0 \quad \text{or} \quad 2\cos(t) + 1 &= 0 \end{aligned}$$

Now, we work on solving the equations individually:

$$\begin{aligned} &\sin(t) = 0 \implies t = 0 \text{ or } t = n\pi \\ &\text{Thus, in the specified interval, we choose } t = 0, \quad t = \pi \end{aligned}$$

As for the next equation:

$$\begin{aligned} 2\cos(t) + 1 &= 0 \\ \cos(t) &= -\frac{1}{2} \end{aligned}$$

We recognize this as a special ratio from a 30-60-90 triangle. Our reference angle is $\frac{\pi}{3}$ but we must be in the second and third quadrants since $\cos(t)$ is negative. This gives us $t_2 = \frac{2\pi}{3} + 2\pi n$ and $t_3 = \frac{4\pi}{3} + 2\pi n$, but in the restricted interval, we find that $t_2 = \frac{2\pi}{3}$ and $t_3 = \frac{4\pi}{3}$.

Finally, our full solution set is $t = \{0, \pi, \frac{2\pi}{3}, \frac{4\pi}{3}\}$.

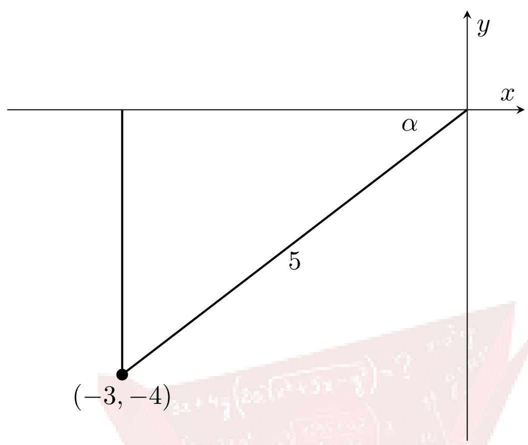
### 13.2 Addition and Subtraction Formulas

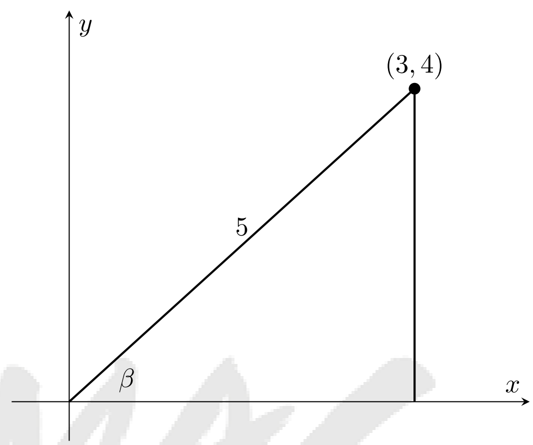
If $\sin(\alpha) = -\dfrac{4}{5}$ and $\sec(\beta) = \dfrac{5}{3}$ for a third-quadrant angle $\alpha$ and a first-quadrant $\beta$, find the following:

## Solution

(a) $\sin(\alpha + \beta)$

So, let's get the triangles we are working with. We can complete the triangles using the Pythagorean theorem.




Now with the formulas and knowledge of which quadrants the angles are in:

$$\begin{aligned}\sin(\alpha+\beta) &= \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)\\ &= \left(-\frac{4}{5}\right)\left(\frac{3}{5}\right) + \left(-\frac{3}{5}\right)\left(\frac{4}{5}\right)\\ &= -\frac{24}{25}.\end{aligned}$$

(b) $\tan(\alpha + \beta)$

$$\begin{aligned}\tan(\alpha+\beta) &= \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}\\ &= \frac{\frac{4}{3}+\frac{4}{3}}{1-\frac{4}{3}\cdot\frac{4}{3}} = \frac{\frac{8}{3}}{1-\frac{16}{9}}\\ &= \frac{\frac{8}{3}}{\frac{-7}{9}} = \frac{8}{3}\cdot\frac{9}{-7} = -\frac{24}{7}\end{aligned}$$

(c) the quadrant containing $\alpha + \beta$

Given that sine of the angle $\alpha+\beta$ is negative as seen in part (a), we know that the angle is either in quadrants 3 or 4. Seeing that tangent of our angle is also negative, $\alpha + \beta$ must be in the fourth quadrant.

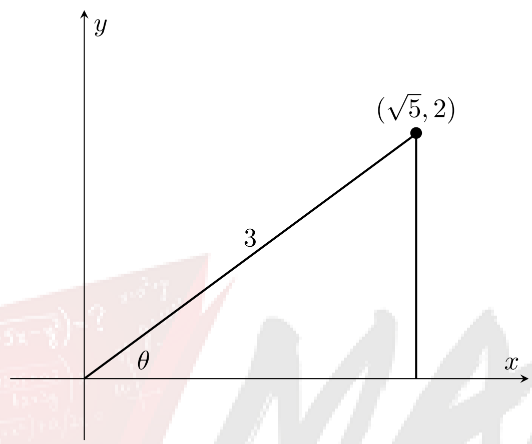
## 13.3 Inverse Trigonometric Functions

Find the exact value whenever it is defined.

## Solution

(a) $\cot\left(\sin^{-1}\dfrac{2}{3}\right)$

Let $\theta = \sin^{-1}\left(\dfrac{2}{3}\right)$ so $\sin\theta = \dfrac{2}{3}$ and we can make our triangle with $\theta$ and complete it using Pythagorean Theorem:



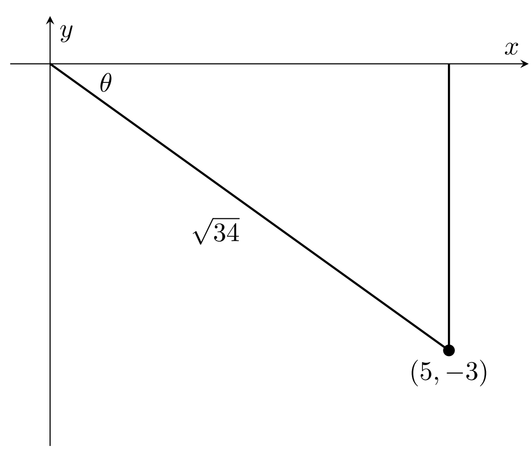

Additionally, $\theta$ can be in the first or fourth quadrants due to the range of the inverse sine function. Since $\dfrac{2}{3} > 0$, we know that $\theta$ is in the first quadrant. Thus:

$$\cot\theta = \frac{\sqrt{5}}{2} \implies \cot\left(\sin^{-1}\frac{2}{3}\right) = \frac{\sqrt{5}}{2}$$



(b) $\sec\left[\tan^{-1} - \left(\dfrac{3}{5}\right)\right]$

Let $\theta = \tan^{-1}\left(-\dfrac{3}{5}\right)$ so $\tan\theta = -\dfrac{3}{5}$ and we can make our triangle with $\theta$ and complete it using Pythagorean Theorem:

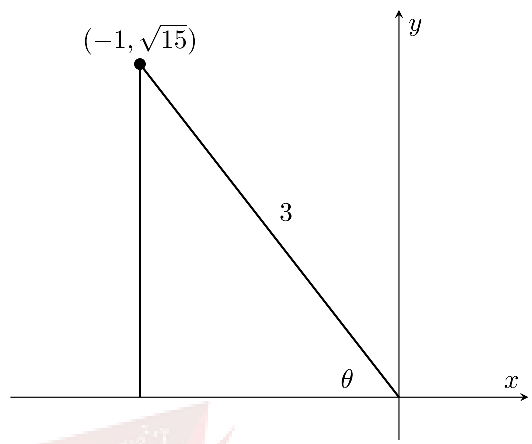


Additionally, $\theta$ can be in the first or fourth quadrants due to the range of inverse tangent. Since $-\frac{3}{5} > 0$, we know that $\theta$ is in the fourth quadrant. Thus:

$$\sec\theta = \frac{\sqrt{34}}{5} \implies \sec\left(\tan^{-1} -\frac{3}{5}\right) = \frac{\sqrt{34}}{5}$$

(c) $\csc\left[\cos^{-1} - \left(\dfrac{1}{4}\right)\right]$

Let $\theta = \cos^{-1}(-\frac{1}{4})$ so $\cos(\theta) = -\frac{1}{4}$ and we can make our triangle with $\theta$ and complete it using pythagorean thm.




Additionally, $\theta$ can be in the first or second quadrants due to the range of inverse cosine. Since $-\frac{1}{4} > 0$, we know that $\theta$ is in the second quadrant. Thus:

$$\csc\theta = \frac{4}{\sqrt{15}} \implies \csc\theta = \frac{4\sqrt{15}}{15}$$
$$\csc\left(\cos^{-1} - \frac{1}{4}\right) = \frac{4\sqrt{15}}{15}$$

Find the exact value whenever it is defined.

## Solution

(a) $\sin\left(\arcsin\left(\frac{1}{2}\right) + \arccos(0)\right)$

We can begin by using the sum formula:

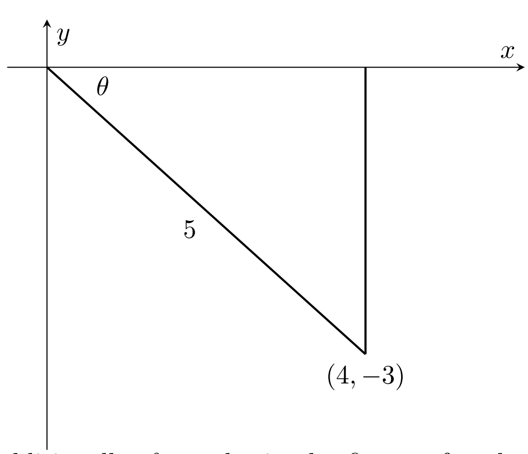
$$\begin{aligned}
&\sin\left(\arcsin\left(\frac{1}{2}\right) + \arccos(0)\right)\\
\Rightarrow &\sin\left(\arcsin\left(\frac{1}{2}\right)\right)\cos(\arccos(0)) + \cos\left(\arcsin\left(\frac{1}{2}\right)\right)\sin(\arccos(0))\\
\Rightarrow &\sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{2}\right)\\
\Rightarrow &\left(\frac{1}{2}\right)(0) + \left(\frac{\sqrt{3}}{2}\right)(1)\\
\Rightarrow &\frac{\sqrt{3}}{2}
\end{aligned}$$

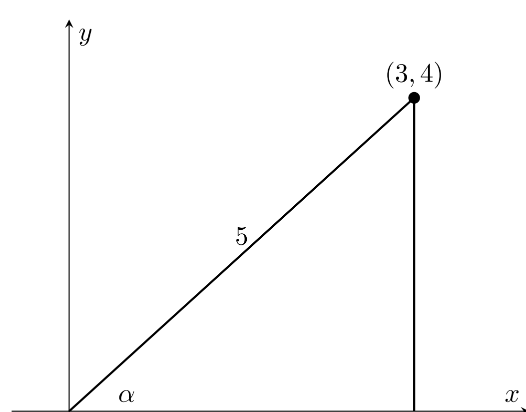
(b) $\cos\left(\arctan\left(-\frac{3}{4}\right) - \arcsin\frac{4}{5}\right)$

Similar to the previous problem, we begin with the sum formula:

$$\begin{aligned}
&\cos\left(\arctan\left(-\frac{3}{4}\right) - \arcsin\frac{4}{5}\right)\\
\Rightarrow &\cos\left(\arctan\left(-\frac{3}{4}\right)\right)\cos\left(\arcsin\left(\frac{4}{5}\right)\right) + \sin\left(\arctan\left(-\frac{3}{4}\right)\right)\sin\left(\arcsin\left(\frac{4}{5}\right)\right)
\end{aligned}$$

Let $\theta = \arctan\left(-\frac{3}{4}\right)$ so $\tan\theta = -\frac{3}{4}$ and we can make our triangle with $\theta$ completing it using Pythagorean Theorem. Let $\alpha = \arcsin\left(\frac{4}{5}\right)$ so $\sin\alpha = \frac{4}{5}$ and we can make our triangle with $\alpha$ completing it using Pythagorean Theorem again as follows:



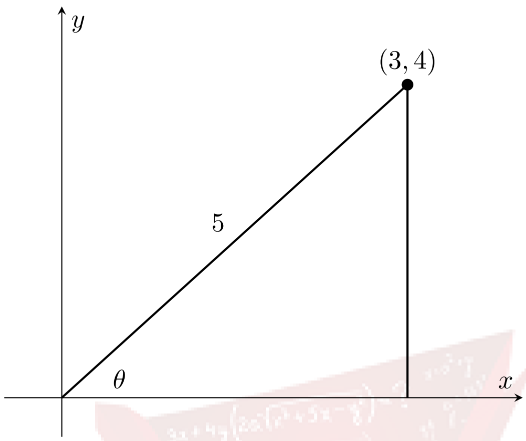

Additionally, $\theta$ can be in the first or fourth quadrants due to the range of inverse tangent. Since

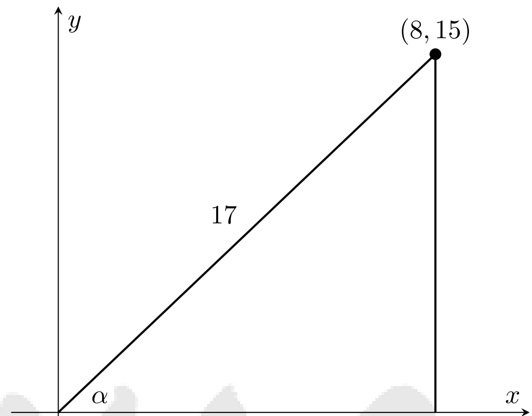
$-\dfrac{3}{4} < 0$, we know that $\theta$ is in the fourth quadrant. $\alpha$ can be in the first or fourth quadrants due to the range of inverse sine. Since $\sin\alpha = \dfrac{4}{5} > 0$, we know that $\alpha$ is in the first quadrant. Now:

$$\begin{aligned}
&\cos\left(\arctan\left(-\frac{3}{4}\right)\right)\cos\left(\arcsin\left(\frac{4}{5}\right)\right)+\sin\left(\arctan\left(-\frac{3}{4}\right)\right)\sin\left(\arcsin\left(\frac{4}{5}\right)\right)\\
&=\left(\frac{4}{5}\right)\left(\frac{3}{5}\right)+\left(-\frac{3}{5}\right)\left(\frac{4}{5}\right)=0
\end{aligned}$$

(c) $\tan\left(\arctan\dfrac{4}{3}+\arccos\dfrac{8}{17}\right)=\dfrac{\tan\left(\arctan\dfrac{4}{3}\right)+\tan\left(\arccos\dfrac{8}{17}\right)}{1-\tan\left(\arctan\dfrac{4}{3}\right)\tan\left(\arccos\dfrac{8}{17}\right)}$

Let $\theta=\arctan\left(\dfrac{4}{3}\right)$ so $\tan\theta=\dfrac{4}{3}$ and we can make our triangle with $\theta$ completing it using Pythagorean Theorem. Let $\alpha=\arccos\left(\dfrac{8}{17}\right)$ so $\cos\alpha=\dfrac{8}{17}$ and we can make our triangle with $\alpha$ completing it using pythagorean thm




Additionally, $\theta$ can be in the first or fourth quadrants due to the range of inverse tangent. Since $\dfrac{4}{3}>0$, we know that $\theta$ is in the first quadrant. $\alpha$ can be in the first or second quadrants due to the range of inverse cosine. Since $\cos\alpha=\dfrac{8}{17}>0$, we know that $\alpha$ is in the first quadrant. Now:

$$
\begin{aligned}
\tan\left(\arctan\frac{4}{3}+\arccos\frac{8}{17}\right) &= \frac{\tan\left(\arctan\frac{4}{3}\right)+\tan\left(\arccos\frac{8}{17}\right)}{1-\tan\left(\arctan\frac{4}{3}\right)\tan\left(\arccos\frac{8}{17}\right)} \\
&= \frac{\frac{4}{3}+\frac{15}{8}}{1-\frac{4}{3}\cdot\frac{15}{8}} = \frac{\frac{77}{24}}{1-\frac{60}{24}} \\
&= \frac{\frac{77}{24}}{\frac{-36}{24}} = \frac{77}{24}\cdot\frac{-24}{36} = -\frac{77}{36}
\end{aligned}
$$

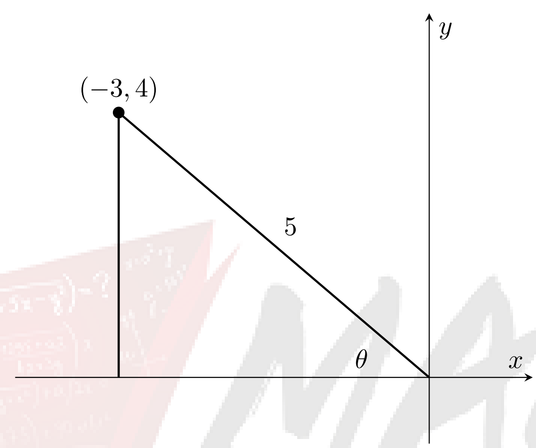
Find the exact value whenever it is defined.

## Solution

(a) $\sin\left[2\arccos\left(-\frac{3}{5}\right)\right] = 2\sin(\arccos\left(-\frac{3}{5}\right))\cos(\arccos\left(-\frac{3}{5}\right))$

Let $\theta = \arccos\left(-\frac{3}{5}\right)$ so $\cos(\theta) = -\frac{3}{5}$ and we can make our triangle with $\theta$ and complete it using Pythagorean Theorem.

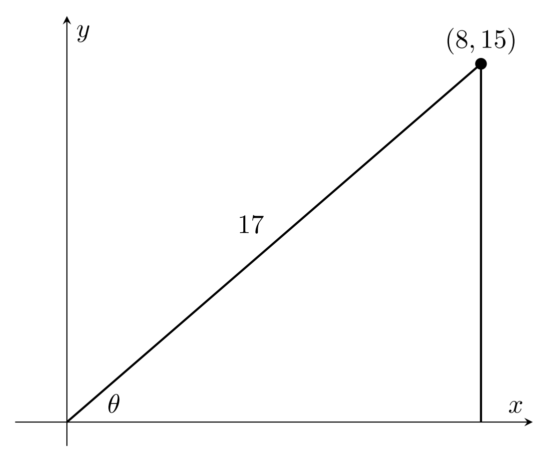
Additionally, $\theta$ can be in the first or second quadrants due to the range of inverse cosine. Since $-\frac{3}{5} < 0$, we know that $\theta$ is in the second quadrant. Thus:




$$2\sin\left[2\arccos\left(-\frac{3}{5}\right)\right] = 2\sin\left(\arccos\left(-\frac{3}{5}\right)\right)\cos\left(\arccos\left(-\frac{3}{5}\right)\right)$$
$$= 2\left(\frac{4}{5}\right)\left(-\frac{3}{5}\right) = -\frac{24}{25}$$

(b) $\cos\left[2\sin^{-1}\left(\frac{15}{17}\right)\right] = \cos^2(\sin^{-1}\left(\frac{15}{17}\right)) - \sin^2(\sin^{-1}\left(\frac{15}{17}\right))$

Let $\theta = \sin^{-1}\left(\frac{15}{17}\right)$ so $\sin(\theta) = \frac{15}{17}$ and we can make our triangle with $\theta$ and complete it using Pythagorean Theorem

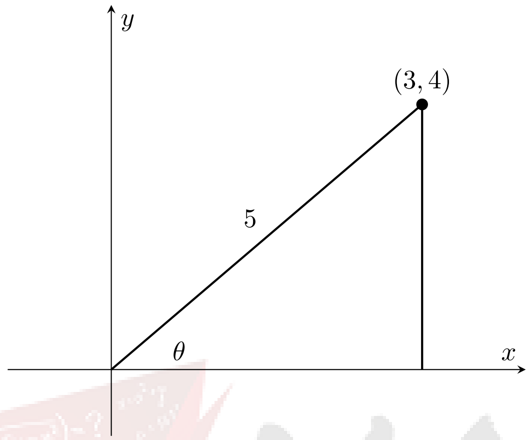


Additionally, $\theta$ can be in the first or fourth quadrants due to the range of inverse sine. Since $\dfrac{15}{17} > 0$, we know that $\theta$ is in the first quadrant. Thus:

$$\begin{aligned}
\cos\left[2\sin^{-1}\left(\frac{15}{17}\right)\right] &= \cos^2\left(\sin^{-1}\left(\frac{15}{17}\right)\right) - \sin^2\left(\sin^{-1}\left(\frac{15}{17}\right)\right) \\
&= \left(\frac{8}{17}\right)^2 - \left(\frac{15}{17}\right)^2 \\
&= \frac{64-225}{289} = -\frac{161}{289}
\end{aligned}$$

(c) $\tan\left[2\tan^{-1}\left(\dfrac{3}{4}\right)\right] = \dfrac{2\tan\left[\tan^{-1}\left(\dfrac{3}{4}\right)\right]}{1-\tan\left[\tan^{-1}\left(\dfrac{3}{4}\right)\right]^2}$

Let $\theta = \tan^{-1}\left(\dfrac{3}{4}\right)$ so $\tan(\theta) = \dfrac{3}{4}$ and we can make our triangle with $\theta$ and complete it using pythagorean thm.



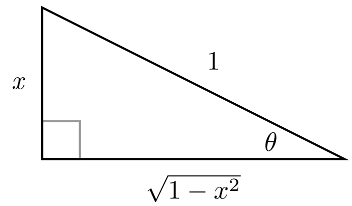

Additionally, $\theta$ can be in the first or fourth quadrants due to the range of inverse tangent. Since $\dfrac{3}{4} > 0$, we know that $\theta$ is in the first quadrant. Thus:

$$
\begin{aligned}
\tan\left[2\tan^{-1}\left(\frac{3}{4}\right)\right] &= \frac{2\tan\left[\tan^{-1}\left(\frac{3}{4}\right)\right]}{1-\tan\left[\tan^{-1}\left(\frac{3}{4}\right)\right]^2} = \frac{2\left(\frac{3}{4}\right)}{1-\left(\frac{3}{4}\right)^2} \\
&= \frac{\frac{6}{4}}{1-\frac{9}{16}} = \frac{6/4}{7/16} \\
&= \frac{6}{4}\cdot\frac{17}{6} = \frac{24}{7}
\end{aligned}
$$

Write the expression as an algebraic expression in $x$ for $x > 0$.

$$\sin(2\sin^{-1} x) = 2\sin(\sin^{-1} x)\cos(\sin^{-1} x)$$

Let $\theta = \sin^{-1} x$ so $\sin(\theta) = \dfrac{x}{1}$ and we can make our triangle with $\theta$ and complete it using Pythagorean Theorem:




Additionally, $\theta$ can be in the first or fourth quadrants due to the range of inverse sine. Since $x > 0$, we know that $\theta$ is in the first quadrant. So:

$$\sin(2\sin^{-1} x) = 2\sin(\sin^{-1} x)\cos(\sin^{-1} x) = 2\left(\frac{x}{1}\cdot\frac{\sqrt{1-x^2}}{1}\right) = 2x\sqrt{1-x^2}$$

# 14 Law of Sines, Law of Cosines, and Heron's Formula

## 14.1 Law of Sines

Solve $\triangle ABC$.

$$\gamma = 81^\circ \qquad c = 11 \qquad b = 12$$

### Solution

Using Law of Sines:

$$\frac{\sin(81^\circ)}{11} = \frac{\sin(\beta)}{12}$$

This gives us

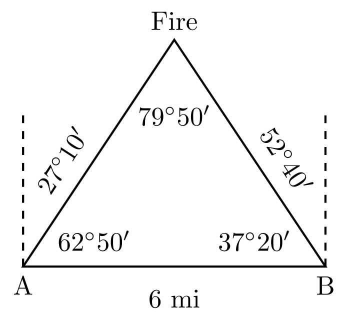
$$\sin(\beta) \approx 1.077 > 1$$

Therefore, no triangle exists with the above conditions.

A forest ranger at an observation point A sights a fire in the direction $N27°10'E$. Another ranger at an observation point B, 6.0 miles due east of A, sight the same fire at $N52°40'W$. Approximate the distance from A to the fire.

## Solution

It is beneficial to draw out the situation, which looks something like the diagram below where the point C is the location of the fire.




From the law of sines, we can see that the following must be true:

$$\frac{\sin(37°20')}{AC} = \frac{\sin(79°50')}{6}$$

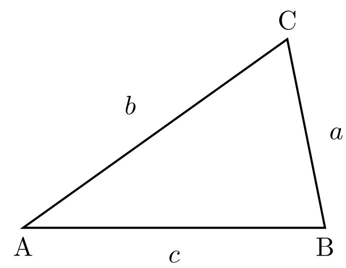
Solving for $AC$ leads to the conclusion that

$$AC = \frac{6\sin(37°20')}{\sin(79°50')} \approx 3.696\text{mi}$$

Solve for the angles in the triangle $\triangle ABC$.

$a = 25.0$ $b = 80.0$ $c = 60.0$

## Solution




Use Law of Cosines to solve for $\alpha$,

$$25^2 = 80^2 + 60^2 - 2(80)(60)\cos(\alpha)$$
$$\frac{125}{128} = \cos(\alpha)$$
$$\alpha = \arccos\left(\frac{125}{128}\right) \approx 12.43^\circ$$

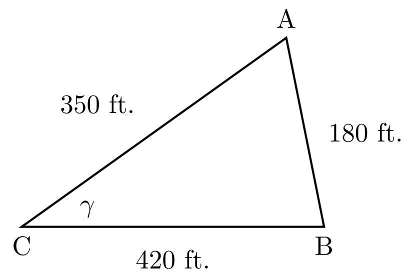
Do the same for $\beta$ and $\gamma$,

$$80^2 = 25^2 + 60^2 - 2(25)(60)\cos(\beta)$$
$$\cos(\beta) \approx -\frac{29}{40}$$
$$\beta \approx 136.47^\circ$$
$$60^2 = 25^2 + 80^2 - 2(25)(80)\cos(\gamma)$$
$$\cos(\gamma) \approx \frac{137}{160}$$
$$\gamma \approx 31.10^\circ$$

A triangular plot of land has sides of lengths 420 feet, 350 feet, and 180 feet. Approximate the smallest angle between the sides.

## Solution

Let $a = 420$, $b = 350$, and $c = 180$. The picture now looks something like the following:




We know that the smallest side is always opposite the smallest angle. The same holds for the largest angle and side. Thus, we need to find $\gamma$. We can use Law of Cosines as follows:

$$c^2 = a^2 + b^2 - 2ab\cos(\gamma)$$

$$180^2 = 420^{+}350^2 - 2(420)(35)\cos(\gamma)$$

$$\cos(\gamma) = \frac{180^2 - 420^2 - 350^2}{-2(420)(350)} \approx 0.906$$

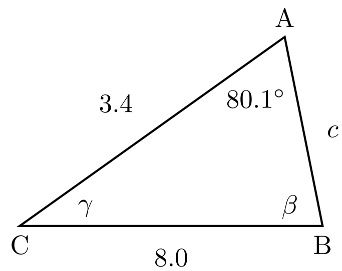
$$\gamma = \cos^{-1}(0.906) \approx 25.04^\circ$$

### 14.2 Law of Sines

Solve for the remaining parts of the triangle $\triangle ABC$.

$$\alpha = 80.1^\circ \qquad a = 8.0 \qquad b = 3.4$$

Again, we begin by drawing out the scenario at hand:




## Solution

Now, we see that we already know an angle the length of its opposite side, which allows us to use the Law of Sines:

$$\begin{aligned}
\frac{\sin\alpha}{a} &= \frac{\sin\beta}{b} \\
\frac{\sin(80.1^\circ)}{8} &= \frac{\sin\beta}{3.4} \\
\sin\beta = \frac{3.4\sin(80.1^\circ)}{8} &\approx 0.4187 \\
\beta = \arcsin(0.4187) &\approx .24.8^\circ
\end{aligned}$$

Now that we have two angles, we can calculate the third angle since all angles within a triangle must add up to $180^\circ$, thus:

$$\begin{aligned}
\gamma &= 180^\circ - \beta - \alpha \\
\gamma &= 180^\circ - 24.8^\circ - 80.1^\circ \\
\gamma &\approx .75.1^\circ
\end{aligned}$$

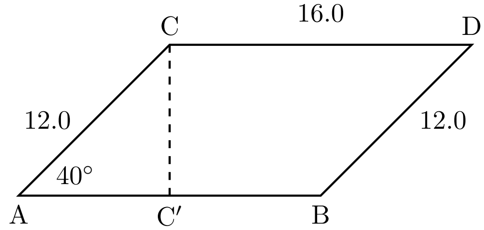
Finally, to determine the side length $c$, we will use the Law of Cosines as follows:

$$\begin{aligned}
c^2 &= a^2 + b^2 - 2ab\cos\gamma \\
c^2 &= (8)^2 + (3.4)^2 - 2(8)(3.4)\cos(75.1^\circ) \\
c^2 &= 61.617 \\
c &= \sqrt{61.617} \approx 7.8
\end{aligned}$$

Approximate the areas of the parallelogram that has sides of length $a$ and $b$ (in feet) if one angle at a vertex has measure $\theta$.

$$a = 12 \text{ ft.} \qquad b = 16 \text{ ft.} \qquad \theta = 40^\circ$$

## Solution




We are aware that the area of a parallelogram can be calculated by simple formula Area = Base × Height. We are provided the base of this triangle, but must calculate the height. We can do so using the trigonometric functions since:

$$\sin\theta = \frac{\text{opp.}}{\text{hyp.}}$$
$$\sin(40^\circ) = \frac{CC'}{AC} = \frac{CC'}{12}$$
$$\text{Height} = CC' = 12\sin(40^\circ) \approx 7.71 \text{ ft.}$$

Thus, the area is going to be:

$$\text{Area} = (16)(7.71) \approx 123.4 \text{ ft.}^2$$

## 14.3 Heron's Formula

Approximate the area of $\triangle ABC$.

$$a = 25.0 \qquad b = 80.0 \qquad c = 60.0$$

## Solution

To begin with, we must calculate the $s$ term as follows:

$$s = \frac{a+b+c}{2} = \frac{25+80+60}{2} = 82.5$$

Next, we apply Heron's formula:

$$\begin{aligned}
\text{Area} &= \sqrt{s(s-a)(s-b)(s-c)} \\
&= \sqrt{(82.5)(82.5-25)(82.5-80)(82.5-60)} \\
&= \sqrt{(82.5)(57.5)(2.5)(22.5)} \\
&= \sqrt{266835.9375} \\
\text{Area} &\approx 516.56 \text{ Units}^2
\end{aligned}$$