Contents

1 Conics

1.1 Parabolas

Find the vertex, focus, and directrix of the parabola. Sketch its graph, showing the focus and the directrix.

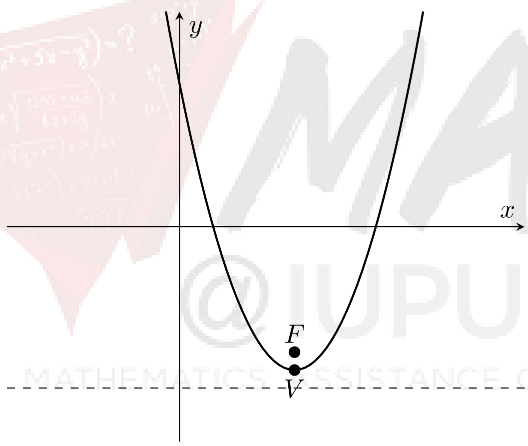
Solution

Begin the problem by completing the square.

$$
\begin{aligned}
y &= x^2 - 4x + 2 \\
y - 2 &= x^2 - 4x \\
y - 2 + 4 &= x^2 - 4x + 4 \\
y + 2 &= (x - 2)^2
\end{aligned}
$$

Now our equation is of the form $4p(y - k) = (x - h)^2$

So the vertex of our parabola $V(h, k)$ is $(2, -2)$. Now $4p = 1$ so $p = \frac{1}{4}$. Our parabola is vertical since x is the squared term. Our focus $F(h, k+p)$ is $(2, -2+\frac{1}{4}) = (2, -\frac{7}{4})$. Lastly, our directrix is $y = k-p = -2-\frac{1}{4} = -\frac{9}{4}$.



Find an equation of the parabola that satisfies the given conditions.

$$\text{Focus } F(6,4), \text{ directrix } y = -2$$

Solution

If our directrix is $y = -2$, then our general equation for the parabola is:

$$4p(y-k) = (x-h)^2$$

And $y = -2 = k - p$. Furthermore, the focus is $F(h, k+p) = (6,4)$ so $h = 6$ and $k + p = 4$. If $k + p = 4$ and $-2 = k - p$, we can find that solve for k and p by adding the two equations:

$$\begin{aligned} &k + p = 4 \\ &k - p = -2 \\ &\implies 2k = 2 \implies k = 1 \end{aligned}$$

Plugging the value of k back in, we see that $p = 3$. Putting all of the pieces together, the equation of the parabola is:

$$12(y-1) = (x-6)^2$$

Find an equation of the parabola that satisfies the given conditions.

Vertex $V(-3,5)$, axis parallel to the x-axis, and passing through the point (5,9)

Solution

If the axis is parallel to the x-axis, then our parabola is horizontal and is of the form:

$$(y-k)^2 = 4p(x-h)$$

The vertex $V(h,k) = (-3,5)$.

Let's plug in the values that we already know.

$$(y-5)^2 = 4p(x+3)$$

We can plug in the additional point we were given so that we can solve for p.

$$(9-5)^2 = 4p(5+3)$$

Solving this equation, we can see that

$$p = \frac{1}{2}$$

So our final equation is

$$(y-5)^2 = 2(x+3)$$

1.2 Ellipses

Find the vertices and foci of the ellipse. Sketch its graph, showing the foci.

$$\frac{(x-3)^2}{16} + \frac{(y+4)^2}{9} = 1$$

Solution

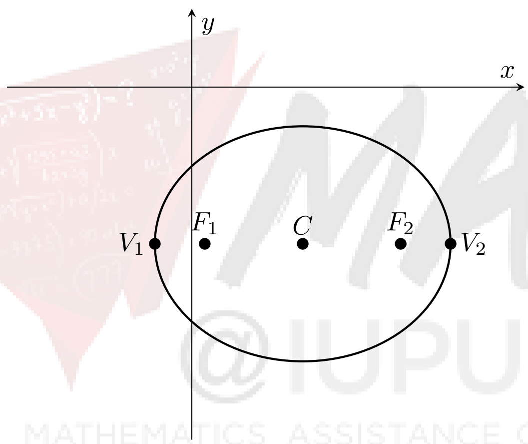
This ellipse is in the form:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

Now we can see that the center $C(h, k) = (3, -4)$ and the major axis runs horizontal away from the center by $a = 4$ in both directions. This gives our vertices as:

$$(3 \pm 4, -4)$$

We know our foci are $(h \pm c, k)$ where $c^2 = a^2 - b^2 = 16 - 9 = 7$. Thus, our foci are $(3 \pm \sqrt{7}, -4)$. See the sketch below.



Find the vertices and foci of the ellipse. Sketch its graph, showing the foci.

$$4x^2 + 9y^2 - 32x - 36y + 64 = 0$$

Solution

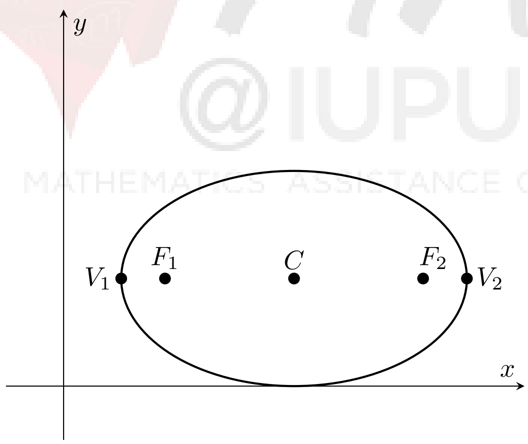
Begin the problem by completing the square for x and y

$$\begin{aligned}
4x^2 + 9y^2 - 32x - 36y + 64 &= 0 \\
4x^2 - 32x + 9y^2 - 36y &= -64 \\
4(x^2 - 8x) + 9(y^2 - 4y) &= -64 \\
4(x^2 - 8x) + 64 + 9(y^2 - 4y) + 36 &= -64 + 64 + 36 \\
4(x^2 - 8x + 16) + 9(y^2 - 4y + 4) &= 36 \\
4(x-4)^2 + 9(y-2)^2 &= 36 \\
\frac{4(x-4)^2}{36} + \frac{9(y-2)^2}{36} &= 1 \\
\frac{(x-4)^2}{9} + \frac{(y-2)^2}{4} &= 1
\end{aligned}$$

By completing the squares, our equation is in the desired form of

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

Our center is $C(h,k) = (4,2)$ and our vertices are $V(h \pm a, k) = (4 \pm 3, 2)$. The foci are $(h \pm c, k)$ where $c^2 = a^2 - b^2 = 9 - 4 = \sqrt{5}$. Thus, our foci are $(4 \pm \sqrt{5}, 2)$.



Find an equation of the ellipse that has its center at the origin and satisfies the given conditions.

$$\text{Vertices } V(\pm 8, 0), \text{ foci } F(\pm 5, 0)$$

Solution

Notice that the $\pm$ being in the x-coordinates tell us that the major axis is horizontal. Then our general equation is:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

With foci $(h \pm c, k)$ and vertices $V(h \pm a, k)$. Studying our vertices and foci, we can see that $h = 0$ and $k = 0$. Furthermore, $a = 8$ and $c = 5$. We can use a and c to find b. If $a^2 - b^2 = c^2$, we see that $b = \sqrt{39}$. Finally, plugging everything in, our equation is :

$$\frac{x^2}{64} + \frac{y^2}{39} = 1$$

1.3 Hyperbolas

Find the vertices, the foci, and the equation of the asymptotes of the hyperbola. Sketch its graph, showing the asymptotes and the foci.

$$4y^2 - x^2 + 40y - 4x + 60 = 0$$

Solution

$$\begin{aligned}
4y^2 - x^2 + 40y - 4x + 60 &= 0\\
4y^2 + 40y - x^2 - 4x &= -60\\
4(y^2 + 10y) - 1(x^2 + 4x) &= -60\\
4(y^2 + 10y) + 100 - 1(x^2 + 4x) - 4 &= -60 + 100 - 4\\
4(y^2 + 10y + 25) - 1(x^2 + 4x + 4) &= 36\\
4(y+5)^2 - (x+2)^2 &= 36\\
\frac{4(y+5)^2}{36} - \frac{(x+2)^2}{36} &= 1\\
\frac{(y+5)^2}{9} - \frac{(x+2)^2}{36} &= 1
\end{aligned}$$

Our hyperbola is in the form:

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

Our hyperbola is centered at $(h, k) = (-2, -5)$ and since y is in the positive term, the hyperbola is vertical. Additionally, we can see that $a = 3$ and $b = 6$. This means that our major axis runs vertically and our vertices are

$$(-2, -5 \pm 3)$$

The foci are found using $(h, k \pm c)$ where $c^2 = a^2 + b^2 = 9 + 36 = 45$ so they become:

$$(-2, -5 \pm 3\sqrt{5})$$

Now, let us find the asymptotes. Using the auxillary rectangle, we know that the asymptotes go through the center and the corners of the rectangle. The corners are $\pm b$ away from the center horizontally and $\pm a$ away vertically. Then the slopes are $\pm\frac{a}{b}$. Using point-slope form, we'll get:

$$\begin{aligned}
y + 5 &= \frac{3}{6}(x+2)\\
y + 5 &= \frac{x}{2} + 1\\
y &= \frac{x}{2} - 4
\end{aligned}$$

$$\begin{aligned}
y + 5 &= -\frac{3}{6}(x+2)\\
y + 5 &= -\frac{x}{2} - 1\\
y &= -\frac{x}{2} - 6
\end{aligned}$$

Find an equation of the hyperbola that has its center at the origin and satisfies the given conditions.

$$\text{Vertices } V(\pm 4, 0), \text{ passing through } (8, 2)$$

Solution

Our parabola is horizontal since its vertices are $\pm$ on the x-axis. This tell us that the general equation is $\dfrac{(x-h)^2}{a^2} - \dfrac{(y-k)^2}{b^2} = 1$ and $V(h \pm a, k) = (0 \pm 4, 0)$ which tell us that $(h, k) = (0, 0)$ and $a = 4$. Now our equation becomes $\dfrac{x^2}{16} - \dfrac{y^2}{b^2} = 1$. To find b, we can use the point that we're given.

$$\begin{aligned} \frac{(8)^2}{16} - \frac{(2)^2}{b^2} &= 1 \\ \frac{64}{16} - \frac{4}{b^2} &= 1 \\ 4 - \frac{4}{b^2} &= 1 \\ -\frac{4}{b^2} &= -3 \\ \frac{4}{b^2} &= 3 \\ \frac{4}{3} &= b^2 \end{aligned}$$

Finally, our equation comes $\dfrac{x^2}{16} - \dfrac{y^2}{\frac{4}{3}} = 1$ which is equivalent to $\dfrac{x^2}{16} - \dfrac{3y^2}{4} = 1$

2 Systems of Equations

2.1 Elimination and Substitution

Use the method of substitution to solve the system.

$$\begin{cases} x^2 + y^2 = 16 \\ 2y - x = 4 \end{cases}$$

Solution

Let's solve the second equation for x giving us $x = 2y - 4$. Now, we can plug it in for x in the first equation and solve for y.

$$\begin{aligned} (2y-4)^2 + y^2 &= 16 \\ 4y^2 - 16y + 16 + y^2 &= 16 \\ 5y^2 - 16y &= 0 \\ y(5y - 16) &= 0 \\ y &= \frac{16}{5}, 0 \end{aligned}$$

Now, we can plug these values into either equation (the second will be easier). **If you plug them into an equation with any terms squared or higher power, you may get extraneous answers. This is fine as long as you double-check each answer gotten this way at the end.**

$$\begin{aligned} 2(\frac{16}{5}) - x &= 4 \\ \frac{32}{5} - x &= 4 \\ -x &= -\frac{12}{5} \\ x &= \frac{12}{5} \end{aligned}$$

Which makes one solutions $\left(\frac{12}{5}, \frac{16}{5}\right)$. Next:

$$\begin{aligned} 2(0) - x &= 4 \\ -x &= 4 \\ x &= -4 \end{aligned}$$

So our second solution is $(-4, 0)$.

Use the method of substitution to solve the system.

$$\begin{cases} y^2 - 4x^2 = 4 \\ 9y^2 + 16x^2 = 140 \end{cases}$$

2.2 Solution

Let's solve the first equation for y^2. Note that I won't solve y as we'll have to square it when we substitute it anyhow. $y = 4x^2 + 4$ and we can substitute this into the second equation.

$$\begin{aligned} 9(4x^2 + 4) + 16x^2 &= 140 \\ 36x^2 + 36 + 16x^2 &= 140 \\ 52x^2 &= 104 \\ x^2 &= 2 \\ x &= \pm\sqrt{2} \end{aligned}$$

The $\pm$ is crucial! Now let's plug both of these values into either equation.

$$\begin{aligned} x &= \sqrt{2} \\ y^2 - 4(\sqrt{2})^2 &= 4 \\ y^2 - 8 &= 4 \\ y^2 &= 12 \\ y &= \pm 2\sqrt{3} \end{aligned}$$

This gives us two solutions $(\sqrt{2}, 2\sqrt{3})$ and $(\sqrt{2}, -2\sqrt{3})$. Now let $x = -\sqrt{2}$:

$$\begin{aligned} y^2 - 4(-\sqrt{2})^2 &= 4 \\ y^2 - 8 &= 4 \\ y^2 &= 12 \\ y &= \pm 2\sqrt{3} \end{aligned}$$

which gives us another two solutions $(-\sqrt{2}, 2\sqrt{3})$ and $(-\sqrt{2}, -2\sqrt{3})$.
Plugging these into either equation can confirm that all four are solutions. **Do not list these solutions as $(\pm 2, \pm 2\sqrt{3})$. This is technically different.** It's best to list out each solution explicitly.

Use the method of substitution to solve the system.

$$\begin{cases} x = y^2 - 4y + 5 \\ x - y = 1 \end{cases}$$

Solving for x in the second equation gives $x = y + 1$. Plug this into the first equation and solve for y.

$$\begin{aligned} (y+1) &= y^2 - 4y + 5 \\ 0 &= y^2 - 5y + 4 \\ (y-1)(y-4) &= 0 \\ y &= 1, 4. \end{aligned}$$

We can plug these into the second equation to find the x values:

$$y = 1 \implies x - (1) = 1 \implies x = 2$$
$$\text{Thus the first solution is } (2, 1)$$

Now for $y = 4$:

$$y = 4 \implies x - (4) = 1 \implies x = 5$$
$$\text{Thus the second solution is } (5, 4)$$

2.3 Applied Problems

The price of admission to a high school play was \$3.00 for students and \$4.50 for nonstudents. If 450 tickets were sold for a total of \$1555.50, how many of each kind were purchased?

Solution

Let x be the number of student tickets sold and y the number of nonstudent tickets sold. We know that $x + y = 450$. Considering each of their costs, we also know that $3(x) + 4.5(y) = 1555.50$. This gives us our system of equations.

$$\begin{cases} x + y = 450 \\ 3x + 4.5y = 1555.50 \end{cases}$$

Solving for x in the first equation, $x = 450 - y$. Now plugging this into the second equation:

$$\begin{aligned} &3(450 - y) + 4.5y = 1555.50 \\ &1350 - 3y + 4.5y = 1555.50 \\ &1.5y = 205.5 \\ &y = 137 \end{aligned}$$

and plugging this back into the first equation, we find that $x = 313$. Thus, 313 student tickets and 137 nonstudent tickets were sold.

A small furniture company manufactures sofas and recliners. Each sofa requires 8 hours of labor and \$60 in materials, while a recliner can be built for \$35 in 6 hours. The company has 340 hours of labor available each week and can afford to buy \$2250 worth of materials. How many recliners and sofas can be produced if all labor hours and all materials must be used?

Solution

Let x be the number of sofas built and y the number of recliners built each week. Knowing how many hours each takes to be built, we can see that given a 340 hour limit each week, $8x + 6y = 340$. Similarly, we know the cost of materials for each and the total money available giving the second equation $60x+35y = 2250$.

$$\begin{cases} 8x + 6y = 340 \\ 60x + 35y = 2250 \end{cases}$$

Either method of solving this system would work. We're going to use elimination here. Particularly, we are going to eliminate y by multiplying the first equation by -35 and the second by 6.

$$\begin{cases} -35(8x + 6y) = -35(340) \\ 6(60x + 35y) = 6(2250) \end{cases}$$

$$\begin{cases} -280x - 210y = -11900 \\ 360x + 210y = 13500 \end{cases}$$

Adding these together, we get $80x = 1600$ and $x = 20$. Plugging this into the first equation:

$$\begin{aligned} &8(20) + 6y = 340 \\ &160 + 6y = 340 \\ &6y = 180 \\ &y = 30. \end{aligned}$$

Therefore, 20 sofas and 30 recliners can be produced.

3 Angles and Speeds

3.1 Arcs and Sectors

Given $s = 7$ cm and $r = 4$ cm, answer the following.

(a) Find the radian and degree measures of the central angle θ subtended by the given arc of length s on a circle of radius r.

Solution

The equation we need here is $s = r\theta$, where s is the subtended arc, r is the radius, and θ is the central angle **IN RADIANS**.
Plugging in the info we were given, $7 = 4\theta$ so $\theta = \frac{7}{4}$ (in radians).

(b) Find the area of the sector determined by θ.

Solution

Now the equation we are looking for is $A = \frac{1}{2}r^2\theta$ where A is the area of the sector, r is the radius, and θ is the central angle **in radians**.
$A = \frac{1}{2}(4 \text{ cm})^2(\frac{7}{4}) = 14 \text{ cm}^2$

3.2 Angular and Linear Speed

Given a radius of 5 in. and 40 rpm, answer the following.

(a) Find the angular speed (in radians per minute).

Solution

The relationship we need is $\frac{s}{t} = r \cdot \frac{\theta}{t}$ where $\frac{s}{t}$ is the linear speed and $\frac{\theta}{t}$ is the angular speed. Note the only difference is a factor or r. Also, note that θ must be in radians. The first question asks us for the angular velocity in radians/min but gives us the angular velocity in rpm. All we need to do is convert knowing that 1 rev. $= 2\pi$ so $\frac{40 \text{ rev.}}{\text{min.}} \cdot \frac{2\pi}{1 \text{ rev.}} = 80\pi \frac{\text{rad.}}{\text{min.}}$.

(b) Find the linear speed of a point on the circumference (in ft/min).

Solution

Now, given the angular velocity in the correct units, all we need to do is multiply by our radius and adjust the units to give us the linear velocity. $80\pi \cdot (5 \text{ in}) \cdot \frac{1 \text{ ft.}}{12 \text{ in.}} = \frac{100\pi}{3} \frac{\text{ft.}}{\text{min.}}$.

4 Values of Trigonometric Functions

4.1 Exact Values

Find the exact value.

(a) $\csc\left(\frac{3\pi}{4}\right)$

Solution

First, we know that $\csc(x) = \frac{1}{\sin(x)}$. We also know that $\csc(\theta) = \pm\csc(\theta_{ref})$. So, first we look at the reference angle for $\theta = \frac{3\pi}{4}$. This angle is in the second quadrant and the remaining angle to the x axis gives us a $\theta_{ref} = \frac{\pi}{4}$. So now $\csc(\frac{3\pi}{4}) = \pm\csc(\frac{\pi}{4})$. We decide the $\pm$ by knowing if the trigonometric function should be positive or negative with the original angle. $\frac{3\pi}{4}$ is in the second quadrant and $\csc(x)$ is positive when $\sin(x)$ is positive so $\csc(\frac{3\pi}{4}) = +\csc(\frac{\pi}{4}) = \frac{1}{\sin(\frac{\pi}{4})} = \frac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2}$.

(b) $\csc\left(-\frac{2\pi}{3}\right)$

Solution

This problem could be done exactly the way part a) was done from the beginning, but we'll run through a different method. This requires knowing even and odd functions. $\csc(x)$ is an odd function meaning that $\csc(-x) = -\csc(x)$. So $\csc(-\frac{2\pi}{3}) = -\csc(\frac{2\pi}{3}$. Now using the same technique in part a) and knowing that $\frac{2\pi}{3}$ is in the second quadrant, we see that $-\csc(\frac{2\pi}{3}) = -(+\csc(\frac{\pi}{3})) = \frac{-1}{\sin(\frac{\pi}{3})} = \frac{-1}{\frac{\sqrt{3}}{2}} = \frac{-2}{\sqrt{3}}$ and rationalized $\frac{-2\sqrt{3}}{3}$.

4.2 Approximate Values

Approximate, to the nearest 0.01 radian, all angles θ in the interval $[0, 2\pi)$ that satisfy the equation.

(a) $\sin(\theta) = 0.4195$

Solution

$\theta = \sin^{-}(0.4195) \approx 0.43 = \theta_1$ which is in the first quadrant. This is one of two angles on our interval. The other angle will be in the second quadrant (the other quadrant where *sin* is positive). The second angle will be $\theta_2 = \pi - 0.43 \approx 2.71$.

(b) $\tan(\theta) = -3.2504$

Solution

$\theta = \tan^{-1}(-3.2504) \approx -1.27$. We know that tan^{-1} has a range of $(-\frac{\pi}{2}, \frac{\pi}{2})$ so this angle must be in the fourth quadrant. This is angle is also not in our interval so we'll need to use its reference angle to find the correct solutions. Since -1.27 is in the fourth quadrant, $\theta_{ref} = 1.27$ and we can use this reference angle to find our solutions. We know *tan* is negative in the second and fourth quadrants. These solutions are $\theta_1 = \pi - 1.27 \approx 1.87$ and $\theta_2 = 2\pi - 1.27 \approx 5.01$.

(c) $\sec(\theta) = 1.7452$

Solution

$\sec(\theta) = \dfrac{1}{\cos(\theta)} = 1.7452$. Then $\cos(\theta) = \dfrac{1}{1.7452} \approx 0.57$. So $\theta = \cos^{-1}(0.57) = 0.96 = \theta_1$. The first angle is in the first quadrant and the other quadrant where cos is positive is in the fourth quadrant so $\theta_2 = 2\pi - 0.96 \approx 5.32$.

4.3 Fundamental Identities

Use the fundamental identities to write the first expression in terms of the second, for any acute angle θ.

$$\cot(\theta),\ \sin(\theta)$$

Solution

$\cot(\theta) = \dfrac{\cos(\theta)}{\sin(\theta)}$. Since $\sin^2(\theta) + \cos^2(\theta) = 1$, we can solve for $\cos(\theta)$ and substitute. We can see that $\cos(\theta) = \pm\sqrt{1 - \sin^2(\theta)}$, but θ is acute so it's in the first quadrant and $\cos(\theta)$ must be positive. Therefore, $\cos(\theta) = \sqrt{1 - \sin^2(\theta)}$. Now, we substitute, $\cot(\theta) = \dfrac{\sqrt{1 - \sin^2(\theta)}}{\sin(\theta)}$.

5 Trigonometric Equations and Identities

5.1 Graphs of Trigonometric Functions

Find the amplitude, period, and phase shift and sketching the graph.

$$y = -2\sin(3x - \pi)$$

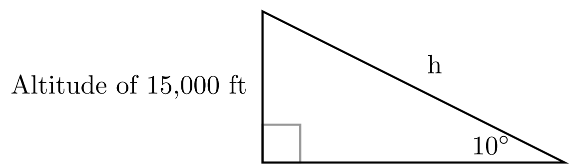
Solution

The standard form for these equations is $y = a\sin(bx + c) + d$, where $|a|$ is amplitude, $\frac{2\pi}{|b|}$ is the period, $\frac{-c}{b}$ is the phase shift, and d is the vertical shift. So, we can see that the amplitude is $|-2| = 2$, the period is $\frac{2\pi}{|3|} = \frac{2\pi}{3}$, and the phase shift is $\frac{-(-\pi)}{3} = \frac{\pi}{3}$. See below for the sketch.

5.2 Applied Problems in Trig.

An airplane takes off at a 10° angle and travels at a rate of 250 ft/sec. Approximately how long does it takes the airplane to reach an altitude of 15,000 feet?

Solution



We can see that there is a trigonometric relationship in this right triangle. $\sin(10°) = \frac{15,000}{h}$. So $h = \frac{15000}{\sin(10°)} \approx 86381.56ft$, but we're not done because the question asked about how long would take to reach the required altitude. Knowing the rate the plane was travelling at, we can calculate this time since:

$$\text{distance} = \text{rate} \cdot \text{time}$$
$$\text{time} = \frac{\text{distance}}{\text{rate}}. \text{ So } t = \frac{86381.56\text{ft.}}{250\frac{\text{ft.}}{\text{sec.}}} \approx 346 \text{ seconds.}$$

5.3 Verifying Identities

Verify the identity.

$$\sec(\theta) - \cos(\theta) = \tan(\theta)\sin(\theta)$$

Solution

First, it's important to note that there are **many ways** to prove these identities. Techniques that should be considered are converting everything to sines and cosines, substituting for squared terms, and making common denominators. For the above identity, let's begin by converting both sides to sines and cosines.

$$\frac{1}{\cos(\theta)} - \cos(\theta) = \frac{\sin(\theta)}{\cos(\theta)}\sin(\theta)$$

Now we can make common denominators on the left side.

$$\begin{aligned}
\frac{1-\cos^2(\theta)}{\cos(\theta)} &= \tan(\theta)\sin(\theta) \\
\frac{\sin^2(\theta)}{\cos(\theta)} &= \tan(\theta)\sin(\theta) \\
\frac{\sin(\theta)}{\cos(\theta)}\sin(\theta) &= \tan(\theta)\sin(\theta) \\
\tan(\theta)\sin(\theta) &= \tan(\theta)\sin(\theta)
\end{aligned}$$

Verify the identity.

$$\frac{1}{1-\cos(\gamma)} + \frac{1}{1+\cos(\gamma)} = 2\csc^2(\gamma)$$

Solution

We can begin by making common denominators on the left side.

$$\frac{1+\cos(\gamma)}{(1-\cos(\gamma)(1+\cos(\gamma))} + \frac{1-\cos(\gamma)}{(1+\cos(\theta)(1-\cos(\gamma))} = 2\csc^2(\gamma)$$

$$\frac{1+\cos(\gamma)+1-\cos(\gamma)}{1-\cos^2(\gamma)} = 2\csc^2(\gamma)$$

but we know that $1-\cos^2(\gamma) = \sin^2(\gamma)$

$$\frac{2}{\sin^2(\gamma)} = 2\csc^2(\gamma)$$

Verify the identity.

$$\tan^4(k) - \sec^4(k) = 1 - 2\sec^2(k)$$

Solution

Fourth powers are hard to work with so we consider how we can reduce them and notice that the left side is a difference of squares so we now have:

$$\begin{aligned}
&(\tan^2(k) - \sec^2(k)) \cdot (\tan^2(k) + \sec^2(k)) = 1 - 2\sec^2(k) \\
&\text{Since } \tan^2(k) + 1 = \sec^2(k) \implies \tan^2(k) - \sec^2(k) = -1 \\
&(-1) \cdot (\tan^2(k) + \sec^2(k)) = 1 - 2\sec^2(k)
\end{aligned}$$

but the right-hand side only has secant in it, so we can subtitute in for the tangent on the left side.

$$\begin{aligned}
&(-1)((\sec^2(k) - 1) + \sec^2(k)) = 1 - 2\sec^2(k) \\
&(-1)(2\sec^2(k) - 1) = 1 - 2\sec^2(k)
\end{aligned}$$

5.4 Finding Solutions of Trig. Equations

Find all solutions to the equation.

$$\sin\left(2x - \frac{\pi}{3}\right) = \frac{1}{2}$$

Solution

First, let us consider $\sin(\theta) = \dfrac{1}{2}$ where $\theta = 2x - \dfrac{\pi}{3}$. We know that θ_{ref} is $\dfrac{\pi}{6}$ from the special 30-60-90 triangle, and there will be two solutions, θ_1 and θ_2, in the two quadrants where sine is positive $+2\pi n$. In quadrant 1, $\theta_1 = \dfrac{\pi}{6}$ and in quadrant 2, $\theta_2 = \dfrac{5\pi}{6}$. Our original question asked about x so we can solve for x now.
So, if $\theta_1 = \dfrac{\pi}{6} + 2\pi n$ then $2x_1 - \dfrac{\pi}{3} = \dfrac{\pi}{6} + 2\pi n$ and we can solve for x_1.

$$\begin{aligned}
2x_1 &= \frac{\pi}{6} + \frac{\pi}{3} + 2\pi n \\
2x_1 &= \frac{\pi}{2} + 2\pi n \\
x_1 &= \frac{\pi}{4} + \pi n.
\end{aligned}$$

Now we can do the same thing with θ_2 to find x_2.

$$\begin{aligned}
2x_2 - \frac{\pi}{3} &= \frac{5\pi}{6} + 2\pi n \\
2x_2 &= \frac{5\pi}{6} + \frac{\pi}{3} + 2\pi n \\
2x_2 &= \frac{7\pi}{6} + 2\pi n \\
x_2 &= \frac{7\pi}{12} + \pi n
\end{aligned}$$

Find the solutions that are in the interval $[0, 2\pi)$.

$$2\tan(t) - \sec^2(t) = 0$$

Solution

To solve this equation, we first need all of the trigonometric functions to be the same. We can make a substitution for $\sec^2(t)$ since we know that:

$$\sec^2(t) = \tan^2(t) + 1.$$

$$\begin{aligned} 2\tan(t) - (\tan^2(t) + 1) &= 0 \\ -\tan^2(t) + 2\tan(t) - 1 &= 0 \\ \tan^2(t) - 2\tan(t) + 1 &= 0 \quad \text{and now using the quadratic equation:} \\ \tan(t) = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(1)}}{2(1)} &= \frac{2 \pm \sqrt{0}}{2} = 1 \end{aligned}$$

We know that if $\tan(t) = 1$, t must be in the first or third quadrant. Additionally, this is a special ratio from a 45-45-90 triangle. Then, we can conclude that **all** solutions would be $t = \frac{\pi}{4} + \pi n$, but our solutions must be in the interval $[0, 2\pi)$, so we can see that $t = \left\{\frac{\pi}{4}, \frac{5\pi}{4}\right\}$.

Approximate, to the nearest 10', the solutions in the interval $[0°, 360°)$.

$$\sin^2(t) - 4\sin(t) + 1 = 0$$

Solution

We begin with the quadratic equation.

$\sin(t) = \dfrac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(1)}}{2(1)} = \dfrac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$ Since these do not relate to any special triangles, we can use inverse trig functions to find the approximate solutions.

Since $\sin(t) \leq 1$ for all t and $1 < 2 + \sqrt{3}$, there are no solutions for this value.

Now, for the other value:

$$\sin(t) = 2 - \sqrt{3}$$
$$t = \sin^{-1}(2 - \sqrt{3}) \approx 15.54° \approx 15°30'$$

Since we are looking for all solutions on $[0°, 360°)$ and we know $\sin^{-1}$ will only give us one of the two, we need to consider the other solution. Since $2 - \sqrt{3} > 0$ and our first solution was in the first quadrant, we need the other quadrant where sine is positive, the second quadrant. So our second solution is $180° - 15°30' = 164°30'$. $t = \{15°30', 164°30'\}$.

6 Inverse Trigonometric Functions And Multiple Angle Formulas

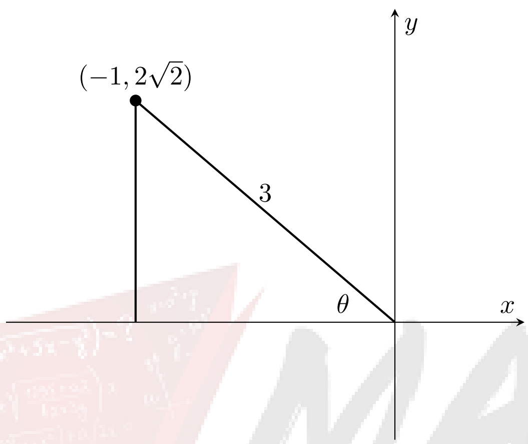
6.1 Double Angle Formulas

Find the exact values of $\sin(2\theta)$, $\cos(2\theta)$, and $\tan(2\theta)$ given the information below.

$$\sec(\theta) = -3, \quad 90^\circ < \theta < 180^\circ$$

Solution

First, we can see that $\cos(\theta) = -\frac{1}{3}$ gives us:



We know that $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ and using the triangle and the fact that θ is in the second quadrant:

$$\sin(2\theta) = 2(\frac{2\sqrt{2}}{3})(-\frac{1}{3}) = \frac{-4\sqrt{2}}{9}$$

$$\cos(2\theta) = 2\cos^2(\theta) - 1 = 2(-\frac{1}{3})^2 - 1 = \frac{2}{9} - 1 = \frac{-7}{9}$$

$$\tan(2\theta) = \frac{2\tan(\theta)}{1-\tan^2(\theta)} = \frac{2(-2\sqrt{2})}{1-(-2\sqrt{2})^2} = \frac{-4\sqrt{2}}{1-8} = \frac{-4\sqrt{2}}{-7} = \frac{4\sqrt{2}}{7}$$

Use inverse trigonometric functions to find the solutions of the equation that are on $[0, 2\pi)$, and approximate solutions to four decimal places.

$$\cos^2(x) + 2\cos(x) - 1 = 0$$

Solution

Using the quadratic formula:

$$\begin{aligned} \cos(x) &= \frac{-(2) \pm \sqrt{(2)^2 - 4(1)(-1)}}{2(1)} \\ &= \frac{-2 \pm \sqrt{8}}{2} = \frac{-2 \pm 2\sqrt{2}}{2} \\ \cos(x) &= -1 \pm \sqrt{2}. \end{aligned}$$

Since these do not relate to any special triangles, we can use inverse trig functions to find the approximate solutions.

$$\begin{aligned} x &= \cos^{-1}(-1 - \sqrt{2}) \quad \text{does not exist since } -1 - \sqrt{2} < -1, \text{ Thus:} \\ x_1 &= \cos^{-1}(-1 + \sqrt{2}) \approx 1.1427 \end{aligned}$$

This is in the first quadrant since $-1 + \sqrt{2} > 0$. Note that $x_1 = x_{\text{ref}}$ since we are in the first quadrant. The other solution we are looking for has the same reference angle and is in the other quadrant where cosine is positive, the fourth quadrant. So, the other solution is

$$x_2 = 2\pi - x_{\text{ref}} = 2\pi - 1.1437 \approx 5.1395$$

Find the solutions that are in the interval $[0, 2\pi)$.

$$\sin(2t) + \sin(t) = 0$$

Solution

Our first step is to rewrite $\sin(2t)$ so that all of the arguments (the inside of the trig functions) are the same. Using a double angle formula, we get:

$$\begin{aligned} 2\sin(t)\cos(t) + \sin(t) &= 0 \\ \sin(t)[2\cos(t) + 1] &= 0 \\ \sin(t) = 0 \quad \text{or} \quad 2\cos(t) + 1 &= 0 \end{aligned}$$

Now, we work on solving the equations individually:

$$\sin(t) = 0 \implies t = 0 \text{ or } t = n\pi$$
$$\text{Thus, in the specified interval, we choose } t = 0, \quad t = \pi$$

As for the next equation:

$$\begin{aligned} 2\cos(t) + 1 &= 0 \\ \cos(t) &= -\frac{1}{2} \end{aligned}$$

We recognize this as a special ratio from a 30-60-90 triangle. Our reference angle is $\frac{\pi}{3}$ but we must be in the second and third quadrants since $\cos(t)$ is negative. This gives us $t_2 = \frac{2\pi}{3} + 2\pi n$ and $t_3 = \frac{4\pi}{3} + 2\pi n$, but in the restricted interval, we find that $t_2 = \frac{2\pi}{3}$ and $t_3 = \frac{4\pi}{3}$.

Finally, our full solution set is $t = \{0, \pi, \frac{2\pi}{3}, \frac{4\pi}{3}\}$.

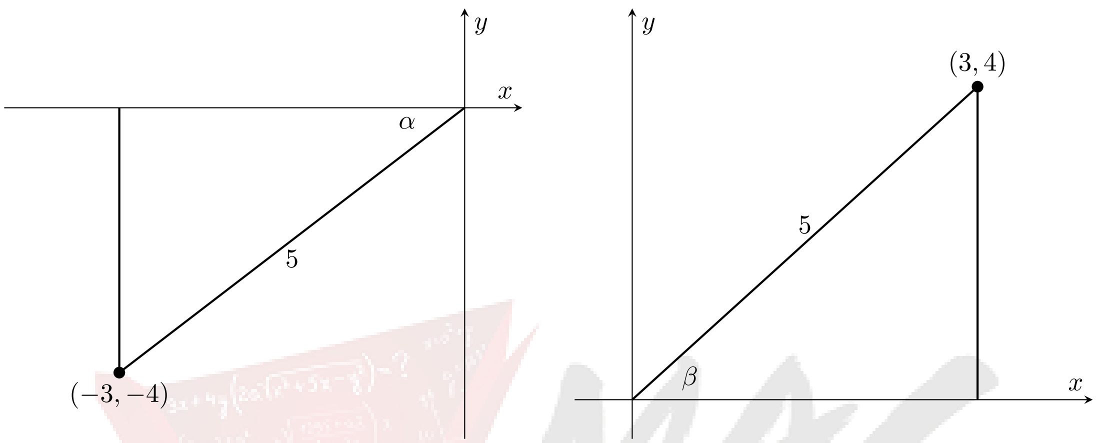
6.2 Addition and Subtraction Formulas

If $\sin(\alpha) = -\dfrac{4}{5}$ and $\sec(\beta) = \dfrac{5}{3}$ for a third-quadrant angle α and a first-quadrant β, find the following:

Solution

(a) $\sin(\alpha+\beta)$

So, let's get the triangles we are working with. We can complete the triangles using the Pythagorean theorem.



Now with the formulas and knowledge of which quadrants the angles are in:

$$\begin{aligned}\sin(\alpha+\beta) &= \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)\\ &= \left(-\frac{4}{5}\right)\left(\frac{3}{5}\right) + \left(-\frac{3}{5}\right)\left(\frac{4}{5}\right)\\ &= -\frac{24}{25}.\end{aligned}$$

(b) $\tan(\alpha+\beta)$

$$\begin{aligned}\tan(\alpha+\beta) &= \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}\\ &= \frac{\frac{4}{3}+\frac{4}{3}}{1-\frac{4}{3}\cdot\frac{4}{3}} = \frac{\frac{8}{3}}{1-\frac{16}{9}}\\ &= \frac{\frac{8}{3}}{\frac{-7}{9}} = \frac{8}{3}\cdot\frac{9}{-7} = -\frac{24}{7}\end{aligned}$$

(c) the quadrant containing $\alpha + \beta$

Given that sine of the angle $\alpha+\beta$ is negative as seen in part (a), we know that the angle is either in quadrants 3 or 4. Seeing that tangent of our angle is also negative, $\alpha + \beta$ must be in the fourth quadrant.

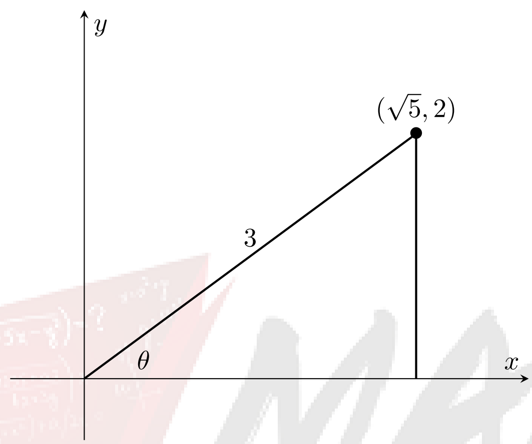
6.3 Inverse Trigonometric Functions

Find the exact value whenever it is defined.

Solution

(a) $\cot\left(\sin^{-1}\frac{2}{3}\right)$

Let $\theta = \sin^{-1}\left(\frac{2}{3}\right)$ so $\sin\theta = \frac{2}{3}$ and we can make our triangle with θ and complete it using Pythagorean Theorem:

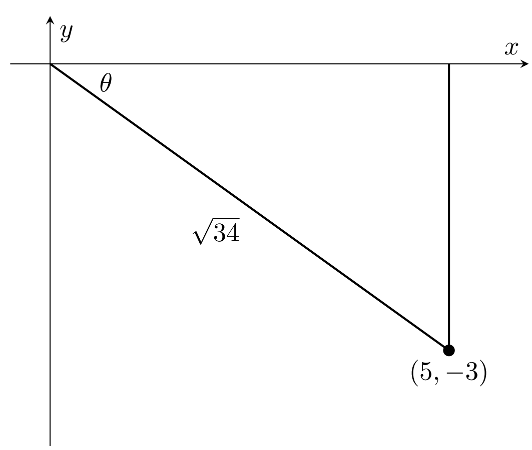


Additionally, θ can be in the first or fourth quadrants due to the range of the inverse sine function. Since $\frac{2}{3} > 0$, we know that θ is in the first quadrant. Thus:

$$\cot\theta = \frac{\sqrt{5}}{2} \implies \cot\left(\sin^{-1}\frac{2}{3}\right) = \frac{\sqrt{5}}{2}$$

(b) $\sec\left[\tan^{-1} - \left(\frac{3}{5}\right)\right]$

Let $\theta = \tan^{-1}\left(-\frac{3}{5}\right)$ so $\tan\theta = -\frac{3}{5}$ and we can make our triangle with θ and complete it using Pythagorean Theorem:

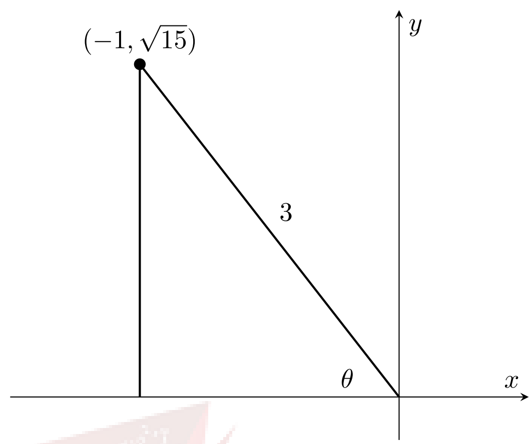


Additionally, θ can be in the first or fourth quadrants due to the range of inverse tangent. Since $-\frac{3}{5} > 0$, we know that θ is in the fourth quadrant. Thus:

$$\sec\theta = \frac{\sqrt{34}}{5} \implies \sec\left(\tan^{-1} -\frac{3}{5}\right) = \frac{\sqrt{34}}{5}$$

(c) $\csc\left[\cos^{-1} - \left(\frac{1}{4}\right)\right]$

Let $\theta = \cos^{-1}(-\frac{1}{4})$ so $\cos(\theta) = -\frac{1}{4}$ and we can make our triangle with θ and complete it using pythagorean thm.



Additionally, θ can be in the first or second quadrants due to the range of inverse cosine. Since $-\frac{1}{4} > 0$, we know that θ is in the second quadrant. Thus:

$$\csc\theta = \frac{4}{\sqrt{15}} \implies \csc\theta = \frac{4\sqrt{15}}{15}$$

$$\csc\left(\cos^{-1} - \frac{1}{4}\right) = \frac{4\sqrt{15}}{15}$$

Find the exact value whenever it is defined.

Solution

(a) $\sin\left(\arcsin\left(\frac{1}{2}\right)+\arccos(0)\right)$

We can begin by using the sum formula:

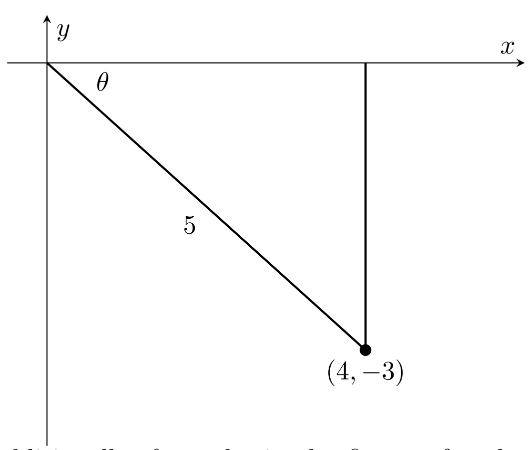
$$\begin{aligned}
&\sin\left(\arcsin\left(\frac{1}{2}\right)+\arccos(0)\right)\\
\Rightarrow&\sin\left(\arcsin\left(\frac{1}{2}\right)\right)\cos(\arccos(0))+\cos\left(\arcsin\left(\frac{1}{2}\right)\right)\sin(\arccos(0))\\
\Rightarrow&\sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{2}\right)+\cos\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{2}\right)\\
\Rightarrow&\left(\frac{1}{2}\right)(0)+\left(\frac{\sqrt{3}}{2}\right)(1)\\
\Rightarrow&\frac{\sqrt{3}}{2}
\end{aligned}$$

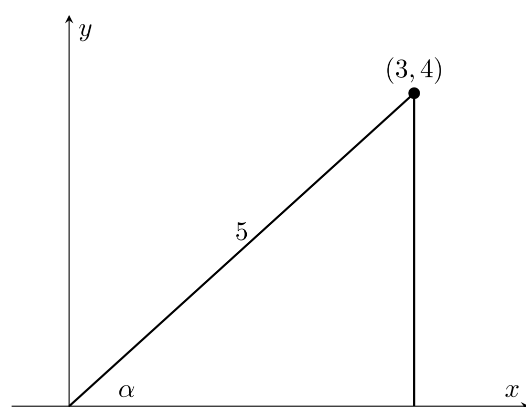
(b) $\cos\left(\arctan\left(-\frac{3}{4}\right)-\arcsin\frac{4}{5}\right)$

Similar to the previous problem, we begin with the sum formula:

$$\begin{aligned}
&\cos\left(\arctan\left(-\frac{3}{4}\right)-\arcsin\frac{4}{5}\right)\\
\Rightarrow&\cos\left(\arctan\left(-\frac{3}{4}\right)\right)\cos\left(\arcsin\left(\frac{4}{5}\right)\right)+\sin\left(\arctan\left(-\frac{3}{4}\right)\right)\sin\left(\arcsin\left(\frac{4}{5}\right)\right)
\end{aligned}$$

Let $\theta=\arctan\left(-\frac{3}{4}\right)$ so $\tan\theta=-\frac{3}{4}$ and we can make our triangle with θ completing it using Pythagorean Theorem. Let $\alpha=\arcsin\left(\frac{4}{5}\right)$ so $\sin\alpha=\frac{4}{5}$ and we can make our triangle with α completing it using Pythagorean Theorem again as follows:





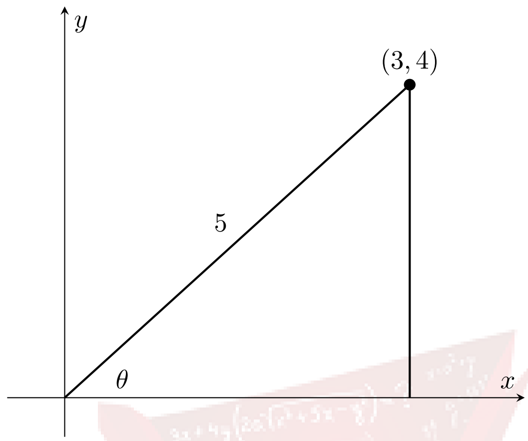
Additionally, θ can be in the first or fourth quadrants due to the range of inverse tangent. Since

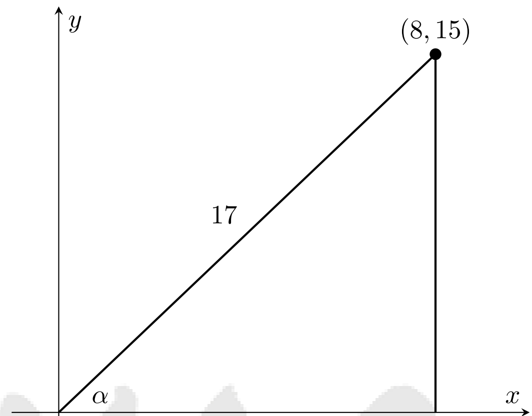
$-\dfrac{3}{4} < 0$, we know that θ is in the fourth quadrant. α can be in the first or fourth quadrants due to the range of inverse sine. Since $\sin\alpha = \dfrac{4}{5} > 0$, we know that α is in the first quadrant. Now:

$$\cos\left(\arctan\left(-\frac{3}{4}\right)\right)\cos\left(\arcsin\left(\frac{4}{5}\right)\right)+\sin\left(\arctan\left(-\frac{3}{4}\right)\right)\sin\left(\arcsin\left(\frac{4}{5}\right)\right)$$
$$=\left(\frac{4}{5}\right)\left(\frac{3}{5}\right)+\left(-\frac{3}{5}\right)\left(\frac{4}{5}\right)=0$$

(c) $\tan\left(\arctan\frac{4}{3}+\arccos\frac{8}{17}\right)=\dfrac{\tan\left(\arctan\frac{4}{3}\right)+\tan\left(\arccos\frac{8}{17}\right)}{1-\tan\left(\arctan\frac{4}{3}\right)\tan\left(\arccos\frac{8}{17}\right)}$

Let $\theta=\arctan\left(\frac{4}{3}\right)$ so $\tan\theta=\frac{4}{3}$ and we can make our triangle with θ completing it using Pythagorean Theorem. Let $\alpha=\arccos\left(\frac{8}{17}\right)$ so $\cos\alpha=\frac{8}{17}$ and we can make our triangle with α completing it using pythagorean thm





Additionally, θ can be in the first or fourth quadrants due to the range of inverse tangent. Since $\frac{4}{3}>0$, we know that θ is in the first quadrant. α can be in the first or second quadrants due to the range of inverse cosine. Since $\cos\alpha=\frac{8}{17}>0$, we know that α is in the first quadrant. Now:

$$\begin{aligned}\tan\left(\arctan\frac{4}{3}+\arccos\frac{8}{17}\right)&=\frac{\tan\left(\arctan\frac{4}{3}\right)+\tan\left(\arccos\frac{8}{17}\right)}{1-\tan\left(\arctan\frac{4}{3}\right)\tan\left(\arccos\frac{8}{17}\right)}\\&=\frac{\frac{4}{3}+\frac{15}{8}}{1-\frac{4}{3}\cdot\frac{15}{8}}=\frac{\frac{77}{24}}{1-\frac{60}{24}}\\&=\frac{\frac{77}{24}}{\frac{-36}{24}}=\frac{77}{24}\cdot\frac{-24}{36}=-\frac{77}{36}\end{aligned}$$

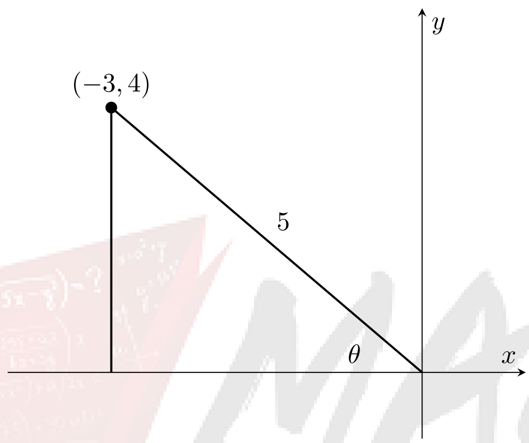
Find the exact value whenever it is defined.

Solution

(a) $\sin\left[2\arccos\left(-\frac{3}{5}\right)\right] = 2\sin(\arccos\left(-\frac{3}{5}\right))\cos(\arccos\left(-\frac{3}{5}\right))$

Let $\theta = \arccos\left(-\frac{3}{5}\right)$ so $\cos(\theta) = -\frac{3}{5}$ and we can make our triangle with θ and complete it using Pythagorean Theorem.

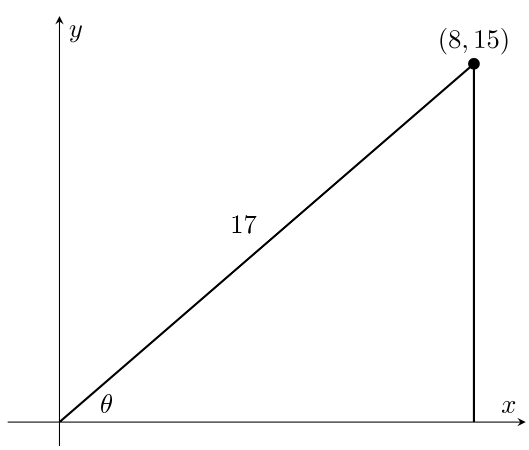
Additionally, θ can be in the first or second quadrants due to the range of inverse cosine. Since $-\frac{3}{5} < 0$, we know that θ is in the second quadrant. Thus:



$$2\sin\left[2\arccos\left(-\frac{3}{5}\right)\right] = 2\sin\left(\arccos\left(-\frac{3}{5}\right)\right)\cos\left(\arccos\left(-\frac{3}{5}\right)\right)$$
$$= 2\left(\frac{4}{5}\right)\left(-\frac{3}{5}\right) = -\frac{24}{25}$$

(b) $\cos\left[2\sin^{-1}\left(\frac{15}{17}\right)\right] = \cos^2(\sin^{-1}\left(\frac{15}{17}\right)) - \sin^2(\sin^{-1}\left(\frac{15}{17}\right))$

Let $\theta = \sin^{-1}\left(\frac{15}{17}\right)$ so $\sin(\theta) = \frac{15}{17}$ and we can make our triangle with θ and complete it using Pythagorean Theorem

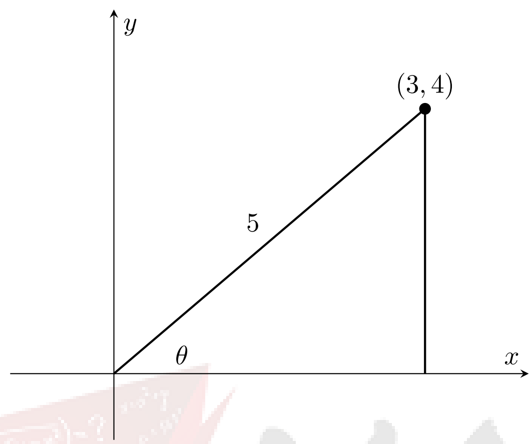


Additionally, θ can be in the first or fourth quadrants due to the range of inverse sine. Since $\frac{15}{17} > 0$, we know that θ is in the first quadrant. Thus:

$$\begin{aligned}\cos\left[2\sin^{-1}\left(\frac{15}{17}\right)\right] &= \cos^2\left(\sin^{-1}\left(\frac{15}{17}\right)\right) - \sin^2\left(\sin^{-1}\left(\frac{15}{17}\right)\right)\\ &= \left(\frac{8}{17}\right)^2 - \left(\frac{15}{17}\right)^2\\ &= \frac{64-225}{289} = -\frac{161}{289}\end{aligned}$$

(c) $\tan\left[2\tan^{-1}\left(\frac{3}{4}\right)\right] = \dfrac{2\tan\left[\tan^{-1}\left(\frac{3}{4}\right)\right]}{1-\tan\left[\tan^{-1}\left(\frac{3}{4}\right)\right]^2}$

Let $\theta = \tan^{-1}\left(\frac{3}{4}\right)$ so $\tan(\theta) = \frac{3}{4}$ and we can make our triangle with θ and complete it using pythagorean thm.



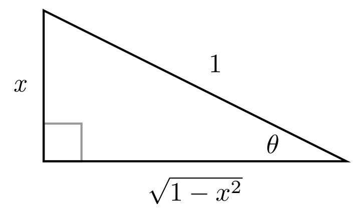
Additionally, θ can be in the first or fourth quadrants due to the range of inverse tangent. Since $\frac{3}{4} > 0$, we know that θ is in the first quadrant. Thus:

$$\begin{aligned}
\tan\left[2\tan^{-1}\left(\frac{3}{4}\right)\right] &= \frac{2\tan\left[\tan^{-1}\left(\frac{3}{4}\right)\right]}{1-\tan\left[\tan^{-1}\left(\frac{3}{4}\right)\right]^2} = \frac{2\left(\frac{3}{4}\right)}{1-\left(\frac{3}{4}\right)^2} \\
&= \frac{\frac{6}{4}}{1-\frac{9}{16}} = \frac{6/4}{7/16} \\
&= \frac{6}{4}\cdot\frac{17}{6} = \frac{24}{7}
\end{aligned}$$

Write the expression as an algebraic expression in x for $x > 0$.

$$\sin(2\sin^{-1}x) = 2\sin(\sin^{-1}x)\cos(\sin^{-1}x)$$

Let $\theta = \sin^{-1}x$ so $\sin(\theta) = \dfrac{x}{1}$ and we can make our triangle with θ and complete it using Pythagorean Theorem:



Additionally, θ can be in the first or fourth quadrants due to the range of inverse sine. Since $x > 0$, we know that θ is in the first quadrant. So:

$$\sin(2\sin^{-1}x) = 2\sin(\sin^{-1}x)\cos(\sin^{-1}x) = 2\left(\frac{x}{1}\cdot\frac{\sqrt{1-x^2}}{1}\right) = 2x\sqrt{1-x^2}$$

7 Law of Sines, Law of Cosines, and Heron's Formula

7.1 Law of Sines

Solve $\triangle ABC$.

$$\gamma = 81^\circ \qquad c = 11 \qquad b = 12$$

Solution

Using Law of Sines:

$$\frac{\sin(81^\circ)}{11} = \frac{\sin(\beta)}{12}$$

This gives us

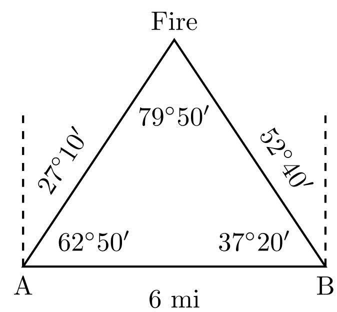
$$\sin(\beta) \approx 1.077 > 1$$

Therefore, no triangle exists with the above conditions.

A forest ranger at an observation point A sights a fire in the direction $N27°10'E$. Another ranger at an observation point B, 6.0 miles due east of A, sight the same fire at $N52°40'W$. Approximate the distance from A to the fire.

Solution

It is beneficial to draw out the situation, which looks something like the diagram below where the point C is the location of the fire.



From the law of sines, we can see that the following must be true:

$$\frac{\sin(37°20')}{AC} = \frac{\sin(79°50')}{6}$$

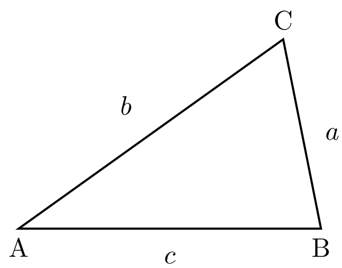
Solving for AC leads to the conclusion that

$$AC = \frac{6\sin(37°20')}{\sin(79°50')} \approx 3.696\text{mi}$$

Solve for the angles in the triangle $\triangle ABC$.

$$a = 25.0 \qquad b = 80.0 \qquad c = 60.0$$

Solution



Use Law of Cosines to solve for α,

$$\begin{aligned} 25^2 &= 80^2 + 60^2 - 2(80)(60)\cos(\alpha) \\ \frac{125}{128} &= \cos(\alpha) \\ \alpha &= \arccos\left(\frac{125}{128}\right) \approx 12.43^\circ \end{aligned}$$

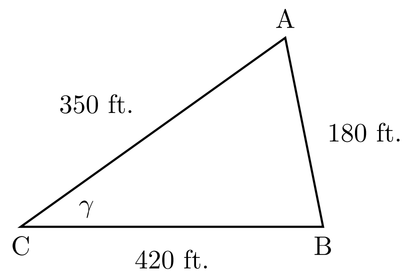
Do the same for β and γ,

$$\begin{aligned} 80^2 &= 25^2 + 60^2 - 2(25)(60)\cos(\beta) \\ \cos(\beta) &\approx -\frac{29}{40} \\ \beta &\approx 136.47^\circ \\ 60^2 &= 25^2 + 80^2 - 2(25)(80)\cos(\gamma) \\ \cos(\gamma) &\approx \frac{137}{160} \\ \gamma &\approx 31.10^\circ \end{aligned}$$

A triangular plot of land has sides of lengths 420 feet, 350 feet, and 180 feet. Approximate the smallest angle between the sides.

Solution

Let $a = 420$, $b = 350$, and $c = 180$. The picture now looks something like the following:



We know that the smallest side is always opposite the smallest angle. The same holds for the largest angle and side. Thus, we need to find γ. We can use Law of Cosines as follows:

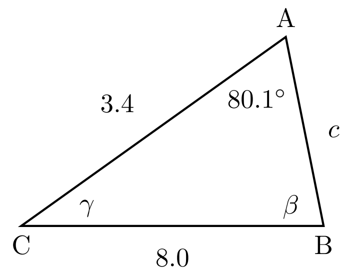
$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos(\gamma) \\ 180^2 &= 420^{+}350^2 - 2(420)(35)\cos(\gamma) \\ \cos(\gamma) &= \frac{180^2 - 420^2 - 350^2}{-2(420)(350)} \approx 0.906 \\ \gamma &= \cos^{-1}(0.906) \approx 25.04^\circ \end{aligned}$$

7.2 Law of Sines

Solve for the remaining parts of the triangle $\triangle ABC$.

$$\alpha = 80.1^\circ \qquad a = 8.0 \qquad b = 3.4$$

Again, we begin by drawing out the scenario at hand:



Solution

Now, we see that we already know an angle the length of its opposite side, which allows us to use the Law of Sines:

$$\begin{aligned} \frac{\sin\alpha}{a} &= \frac{\sin\beta}{b} \\ \frac{\sin(80.1^\circ)}{8} &= \frac{\sin\beta}{3.4} \\ \sin\beta = \frac{3.4\sin(80.1^\circ)}{8} &\approx 0.4187 \\ \beta = \arcsin(0.4187) &\approx .24.8^\circ \end{aligned}$$

Now that we have two angles, we can calculate the third angle since all angles within a triangle must add up to 180°, thus:

$$\begin{aligned} \gamma &= 180^\circ - \beta - \alpha \\ \gamma &= 180^\circ - 24.8^\circ - 80.1^\circ \\ \gamma &\approx .75.1^\circ \end{aligned}$$

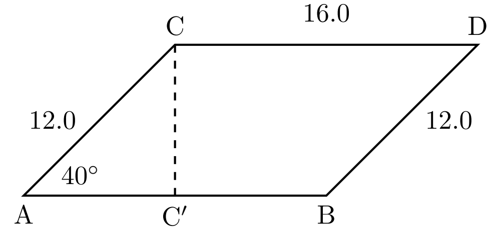
Finally, to determine the side length c, we will use the Law of Cosines as follows:

$$\begin{aligned} c^2 &= a^2 + b^2 - 2ab\cos\gamma \\ c^2 &= (8)^2 + (3.4)^2 - 2(8)(3.4)\cos(75.1^\circ) \\ c^2 &= 61.617 \\ c &= \sqrt{61.617} \approx 7.8 \end{aligned}$$

Approximate the areas of the parallelogram that has sides of length a and b (in feet) if one angle at a vertex has measure θ.

$$a = 12 \text{ ft.} \qquad b = 16 \text{ ft.} \qquad \theta = 40^\circ$$

Solution



We are aware that the area of a parallelogram can be calculated by simple formula Area = Base × Height. We are provided the base of this triangle, but must calculate the height. We can do so using the trigonometric functions since:

$$\sin\theta = \frac{\text{opp.}}{\text{hyp.}}$$
$$\sin(40^\circ) = \frac{CC'}{AC} = \frac{CC'}{12}$$
$$\text{Height} = CC' = 12\sin(40^\circ) \approx 7.71 \text{ ft.}$$

Thus, the area is going to be:

$$\text{Area} = (16)(7.71) \approx 123.4 \text{ ft.}^2$$

7.3 Heron's Formula

Approximate the area of $\triangle ABC$.

$$a = 25.0 \qquad b = 80.0 \qquad c = 60.0$$

Solution

To begin with, we must calculate the s term as follows:

$$s = \frac{a+b+c}{2} = \frac{25+80+60}{2} = 82.5$$

Next, we apply Heron's formula:

$$\begin{aligned}
\text{Area} &= \sqrt{s(s-a)(s-b)(s-c)} \\
&= \sqrt{(82.5)(82.5-25)(82.5-80)(82.5-60)} \\
&= \sqrt{(82.5)(57.5)(2.5)(22.5)} \\
&= \sqrt{266835.9375} \\
\text{Area} &\approx 516.56 \text{ Units}^2
\end{aligned}$$