Contents

1 Real Numbers, Exponents, and Radicals

1.1 Rationalizing the Denominator

Simplify and rationalize the denominator when appropriate.

$$\sqrt[4]{\frac{5x^8y^3}{27x^2}}$$

Solution

First simplify the expression

$$\sqrt[4]{\frac{5\cancel{x^2}x^6y^3}{3^3\cancel{x^2}}} = \sqrt[4]{\frac{5x^6y^3}{3^3}}$$

Rationalize the denominator so that it only contains exponents that are multiples of 4

$$\frac{\sqrt[4]{5x^6y^3}}{\sqrt[4]{3^3}} \cdot \frac{\sqrt[4]{3}}{\sqrt[4]{3}} = \sqrt[4]{\frac{15x^6y^3}{3^4}}$$

Simply both the denominator and numerator

$$\sqrt[4]{\frac{15x^6y^3}{3^4}} = \frac{x}{3}\sqrt[4]{15x^2y^3}$$

1.2 Factoring Polynomials

(a) $64x^3 - y^6$

(b) $y^2 - x^2 + 8y + 16$

Solution

(a) Recognize this as a difference of cubes

$$(4x)^3 - (y^2)^3$$

Use the difference of cubes formula to factor

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$
$$(4x)^3 - (y^2)^3 = (4x - y^2)(16x^2 + 4xy^2 + y^4)$$

(b) Rearrange to form a perfect square

$$y^2 + 8y + 16 - x^2$$

$$y^2 + 8y + 16 - x^2 = (y + 4)^2 - x^2$$

Recognize this as a difference of squares

$$a^2 - b^2 = (a + b)(a - b)$$
$$(y + 4)^2 - x^2 = (y + 4 + x)(y + 4 - x)$$

1.3 Algebraic and Fractional Expressions

Simplify the following expression

$$\frac{(4x^2+9)^{1/2}(2)-(2x+3)\left(\frac{1}{2}\right)(4x^2+9)^{-1/2}(8x)}{[(4x^2+9)^{1/2}]^2}$$

Solution

Begin by factoring out the least common multiple, in this case $(4x^2+9)^{-1/2}$.

$$\frac{(4x^2+9)^{-1/2}\left[(4x^2+9)(2)-(2x+3)\left(\frac{1}{2}\right)(8x)\right]}{\left[\ (4x^2+9)^{1/2}\right]^2}$$

Then reduce exponents

$$\frac{(4x^2+9)(2)-(2x+3)\left(\frac{1}{2}\right)(8x)}{(4x^2+9)^{3/2}}$$

Finally distribute and combine terms in the numerator

$$\begin{aligned} &\frac{8x^2+18-(8x^2+12x)}{(4x^2+9)^{3/2}} \\ &= \frac{\cancel{8x^2}+18\cancel{-8x^2}-12x}{(4x^2+9)^{3/2}} \\ &= \frac{18-12x}{(4x^2+9)^{3/2}} \\ &= \frac{6(3-2x)}{(4x^2+9)^{3/2}} \end{aligned}$$

1.4 Equations

(a) Solve for the specified variable

$$S = \frac{p}{q + p(1-q)} \quad \text{for } q$$

(b) Solve the equation

$$\frac{2}{2x+1} - \frac{3}{2x-1} = \frac{-2x+7}{4x^2-1}$$

(c) Solve for x

$$x = 4 + \sqrt{4x - 19}$$

Solution

(a) Begin by multiplying the denominator over to the left hand side

$$S = \frac{p}{q + p(1-q)}$$
$$S(q + p(1-q)) = p$$

Distribute S and p

$$S(q + p(1-q)) = p$$
$$Sp + Sq - Spq = p$$

Isolate q and solve

$$Sp + Sq - Spq = p$$
$$Sq - Spq = p - Sp$$
$$q(S - Sp) = p - Sp$$
$$q = \frac{p - Sp}{S - Sp}$$
$$q = \frac{p(1-S)}{S(1-p)}$$

(b) First find the Least Common Denominator (LCD) and then multiply the entire equation by it

$$\frac{2}{2x+1} - \frac{3}{2x-1} = \frac{-2x+7}{(2x-1)(2x+1)}$$

$$\left(\frac{2}{2x+1} - \frac{3}{2x-1} = \frac{-2x+7}{(2x-1)(2x+1)}\right)(2x-1)(2x+1)$$

$$\frac{2(2x-1)\cancel{(2x+1)}}{\cancel{2x+1}} - \frac{3\cancel{(2x-1)}(2x+1)}{\cancel{2x-1}} = \frac{-2x+7\cancel{(2x-1)(2x+1)}}{\cancel{(2x-1)(2x+1)}}$$

$$2(2x-1) - 3(2x+1) = -2x + 7$$

Next combine like terms and solve for x

$$4x - 2 - 6x - 3 = -2x + 7$$
$$4x + 2x - 6x = 7 + 3 + 2$$
$$0 \neq 12$$

Since zero is not equal to twelve the equation has **no solutions**.

(c) Begin by isolating the square root

$$x = 4 + \sqrt{4x - 19}$$
$$x - 4 = \sqrt{4x - 19}$$

Square both sides to eliminate the square root and F.O.I.L

$$(x - 4)^2 = (\sqrt{4x - 19})^2$$
$$x^2 - 8x + 16 = 4x - 19$$

Move everything to the left hand side and combine terms

$$x^2 - 8x - 4x + 16 + 19 = 0$$
$$x^2 - 12x + 35 = 0$$

Factor and solve for x

$$(x - 7)(x - 5) = 0$$

$$(x - 7) = 0 \quad \text{and} \quad (x - 5) = 0$$
$$x = 7 \quad \text{and} \quad x = 5$$

NOTE: Don't forget to check your answers when squaring both sides. In this particular problem, $x = 5$ and $x = 7$ are both solutions to the equation.

1.5 Applied Problems

In a certain medical test designed to measure carbohydrate tolerance, an adult drinks 7 ounces of a 30% glucose solution. When the test is administered to a child, the glucose concentration must be decreased to 20%. How much 30% glucose solution and how much water should be used to prepare 7 ounces of 20% glucose solution?

Solution

In order to solve this problem, it is necessary to visualize the problem first.

Children Solution	=	Adult Solution	+	Water
20% glucose	=	30% glucose	+	0% glucose
7 ounces	=	x	+	(7-x) ounces

From the table above, set up the equation.

$$(0.20)(7) = (0.30)(x) + (0.00)(7 - x)$$

Simplify the equation

$$1.4 = 0.3x$$

Solve for x

$$x = \frac{1.4}{0.3}$$

$$x = \frac{14}{3}$$

$\frac{14}{3}$ is the amount of 30% solution needed to prepare 7 ounces of 20% glucose solution.

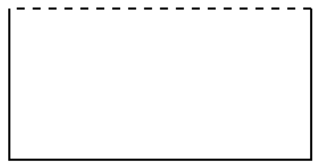
Now calculate the water amount needed.

$$\begin{aligned} \text{water} &= 7 - x \\ &= 7 - \frac{14}{3} \\ &= \frac{21}{3} - \frac{14}{3} \\ \text{water} &= \frac{7}{3} \text{ ounces} \end{aligned}$$

A farmer plans to close a rectangular region, using part of his barn for one side and fencing for the other three sides. If the side parallel to the barn is to be twice the length of the adjacent side, and the area of the region is to be $128ft^2$, how many feet of fencing should be purchased?

Solution

For most word problems, it is important to visualize the problem/situation.



$$\text{Area of rectangle} = (2x)(x)$$
$$\text{Perimeter of rectangle} = 2x + 2x$$

Now that you have a visual, set up the equation.

$$(2x)(x) = 128$$

Simplify the equation and solve for x.

$$2x^2 = 128$$
$$x^2 = \frac{128}{2}$$
$$x^2 = 64$$
$$\sqrt{x^2} = \sqrt{64}$$
$$x = \pm 8$$

Since distances can only be positive, exclude the negative answer. Once you have calculated the width x, calculate the perimeter as follows. Note that the barn is acting as one of the sides and does not need to be accounted for:

$$\text{Perimeter} = 2x + x + x$$
$$= 4x$$
$$= 4(8)$$
$$\text{Perimeter} = 32 \text{ ft.}$$

2 Quadratic Equations and Complex Numbers

2.1 Quadratic Equations

(a) Solve by completing the square

$$4x^2 - 12x - 11 = 0$$

(b) Solve the equation

$$\frac{3}{2}z^2 - 4z - 1 = 0$$

Solution

(a) Begin by dividing equation by 4 in order to get a coefficient of 1 in front of x^2

$$4x^2 - 12x - 11 = 0$$
$$x^2 - 3x - \frac{11}{4} = 0$$

Add $\frac{11}{4}$ to both sides:

$$x^2 - 3x - \cancel{\frac{11}{4}} + \cancel{\frac{11}{4}} = \frac{11}{4}$$

Complete the square as follows:

$$x^2 - 3x + \left(\frac{b}{2}\right)^2 = \frac{11}{4} + \left(\frac{b}{2}\right)^2$$
$$x^2 - 3x + \left(\frac{3}{2}\right)^2 = \frac{11}{4} + \left(\frac{3}{2}\right)^2$$
$$x^2 - 3x + \frac{9}{4} = \frac{11}{4} + \frac{9}{4}$$
$$\left(x - \frac{3}{2}\right)^2 = \frac{20}{4}$$
$$\left(x - \frac{3}{2}\right)^2 = 5$$

We can solve for x by taking the square root of both sides:

$$\sqrt{\left(x - \frac{3}{2}\right)^2} = \pm\sqrt{5}$$
$$x - \frac{3}{2} = \pm\sqrt{5}$$

Finally, Add $\frac{3}{2}$ to both sides to isolate x:

$$x - \cancel{\frac{3}{2}} + \cancel{\frac{3}{2}} = \frac{3}{2} \pm \sqrt{5}$$
$$x = \frac{3}{2} \pm \sqrt{5}$$

(b) We can solve this problem by either completing the square or quadratic formula.

Remember: Quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Begin the problem by multiplying by 2 to get rid of the fraction

$$2\left(\frac{3}{2}z^2 - 4z - 1\right) = (2)(0)$$
$$3z^2 - 8z - 2 = 0$$

We will solve for x using the quadratic formula where $a = 3$, $b = -8$ and $c = -2$.

$$\begin{aligned} z &= \frac{8 \pm \sqrt{8^2 - 4(3)(-2)}}{2(3)} \\ &= \frac{8 \pm \sqrt{64 + 24}}{6} \\ &= \frac{8 \pm \sqrt{88}}{6} \end{aligned}$$

Try to simplify the radical

$$\begin{aligned} z &= \frac{8 \pm \sqrt{88}}{6} \\ &= \frac{8 \pm \sqrt{4 \cdot 22}}{6} \\ &= \frac{8 \pm 2\sqrt{22}}{6} \end{aligned}$$

Thus

$$z = \frac{4}{3} \pm \frac{\sqrt{22}}{3}$$

2.2 Complex Numbers

(a) Write the following expression in the form $a + bi$, where a and b are real numbers.

$$\frac{-4+6i}{2+7i}$$

(b) Find the values of x and y, where x and y are real numbers.

$$(2x - y) - 16i = 10 + 4yi$$

(c) Find the solutions to the equation

$$4x^4 + 25x^2 + 36 = 0$$

Solution

(a) When simplifying fractions, it is important to always rationalize the denominator. Begin the problem by multiplying by the conjugate of the denominator and F.O.I.L.

$$\frac{-4+6i}{2+7i} \cdot \frac{2-7i}{2-7i}$$
$$= \frac{(-4+6i)(2-7i)}{(2+7i)(2-7i)}$$
$$= \frac{-8+28i+12i-42i^2}{4-49i^2}$$

Simplify numerator and denominator.
Remember

i	$\sqrt{-1}$
i^2	-1
i^3	$-\sqrt{-1}$
i^4	1

$$\frac{-8+28i+12i-42(-1)}{4-49(-1)}$$
$$= \frac{-8+28i+12i+42}{4+49}$$
$$= \frac{34+40i}{53}$$

Write in $a \pm bi$ form.

$$= \frac{34}{53} + \frac{40}{53}i$$

(b) The easiest way to solve this problem is by equate the real and imaginary parts

$$(2x - y) - 16i = 10 + 4yi$$

$$\begin{cases} (2x - y) = 10 & (1) \\ -16i = 4yi & (2) \end{cases}$$

First solve for y in equation (2).

$$\begin{aligned} -16\not{i} &= 4y\not{i} \\ -16 &= 4y \\ y &= -4 \end{aligned}$$

Substitute y in equation (1) and solve for x.

$$\begin{aligned} 2x - y &= 10 \\ 2x - (-4) &= 10 \\ 2x + 4 &= 10 \\ 2x &= 10 - 4 \\ x &= \frac{6}{2} \\ x &= 3 \end{aligned}$$

The values of x and y are 3 and -4 , respectively.

(c) In order to solve this problem it is important to recognize it as a quadratic problem.
Start by letting $u = x^2$:

$$\begin{aligned} 4x^4 + 25x^2 + 36 &= 0 \\ 4u^2 + 25u + 36 &= 0 \end{aligned}$$

From here we can factor by using AC-method (a.k.a factor by grouping) or quadratic formula (we will use the AC-method here):

$$\begin{aligned} 4u^2 + 25u + 36 &= 0 \\ 4u^2 + 16u + 9u + 36 &= 0 \\ 4u(u + 4) + 9(u + 4) &= 0 \\ (4u + 9)(u + 4) &= 0 \end{aligned}$$

Set both u factors equal to zero and solve for u:

$$(4u + 9) = 0 \qquad\qquad (u + 4) = 0$$

$$u = \frac{-9}{4} \qquad\qquad u = -4$$

Substitute x^2 back in for u:

$$u = \frac{-9}{4} \qquad\qquad u = -4$$

$$x^2 = \frac{-9}{4} \qquad\qquad x^2 = -4$$

Solve for x by taking the square root of both sides:

$$\sqrt{x^2} = \pm\sqrt{\frac{-9}{4}} \qquad \sqrt{x^2} = \pm\sqrt{-4}$$

$$x = \pm\frac{3}{2}i \qquad x = \pm 2i$$

Finally, write into $a \pm bi$ form:

$$x = 0 \pm \frac{3}{2}i \qquad x = 0 \pm 2i$$

2.3 Applied Problems

A baseball is thrown straight upward with an initial speed of 64 $\frac{\text{ft.}}{\text{sec}}$. The number of feet s above the ground after t seconds is given by the equation

$$s = -16t^2 + 64t$$

(i) When will the baseball be 48 feet above the ground?

(ii) When will it hit the ground?

Solution

(i) To determine when the baseball will be at a height of 48 feet, plug $s = 48$ into the given equation

$$\begin{aligned} 48 &= -16t^2 + 64t \\ 0 &= 16t^2 - 64t + 48 \\ 0 &= 16(t^2 - 4t + 3) \\ 0 &= 16(t-3)(t-1) \\ \Longrightarrow\ t &= 3, 1 \end{aligned}$$

(ii) Ground level corresponds to $s = 0$, plugging that in yields:

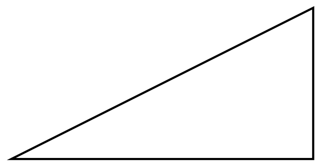
$$\begin{aligned} -16t^2 + 64t &= 0 \\ -16(t^2 - 4t) &= 0 \\ \Longrightarrow\ t &= 0, 4 \end{aligned}$$

The $t = 0$ case corresponds to the point right before the ball was thrown, hence the answer we are interested in is $t = 4$.

The recommended distance d that a ladder should be placed away from a vertical wall is 25% of its length L. Approximate the height h that can be reached by relating h as a percentage of L.

Solution

In this scenario, it is useful to get a picture of what is going on. Draw a right triangle where the base is $L/4$, the hypotenuse is L, and the vertical leg is h as follows:



$$\frac{L^2}{16} + h^2 = L^2$$
$$h^2 = \frac{15L^2}{16}$$
$$h = \frac{\sqrt{15}L}{4} \approx .968L$$

Thus, the approximate height is 97% of L

2.4 Other Types of Equations

Solve the equation

$$2x^{-\frac{2}{3}} - 7x^{-\frac{1}{3}} - 15 = 0$$

Solution

In order to begin this problem, we need to use u-substitution. Begin by substituting u for $x^{-\frac{1}{3}}$:

$$2x^{-\frac{2}{3}} - 7x^{-\frac{1}{3}} - 15 = 0$$
$$2u^2 - 7u - 15 = 0$$

Next solve for u using your favorite method. We will be factoring and solving for u by setting both factors equal to zero:

$$(2u+3)(u-5) = 0$$

$$2u + 3 = 0 \qquad u - 5 = 0$$
$$u = -\frac{3}{2} \qquad u = 5$$

Now we can substitute $x^{-\frac{1}{3}}$ back in for u:

$$x^{-\frac{1}{3}} = -\frac{3}{2} \qquad x^{-\frac{1}{3}} = 5$$

Finally, we can solve for x by raising both sides to the negative third power:

$$(x^{-\frac{1}{3}})^{-3} = \left(-\frac{3}{2}\right)^{-3} \qquad (x^{-\frac{1}{3}})^{-3} = (5)^{-3}$$

Remember that negative exponents, flip coefficients and variables to either the top or bottom and become positive exponents:

$$x = \left(-\frac{2}{3}\right)^3 \qquad x = \left(\frac{1}{5}\right)^3$$

$$x = -\frac{8}{27} \qquad x = \frac{1}{125}$$

3 Inequalities

3.1 Absolute Values

Solve the equation for x.

$$3|x+1|-2=-11$$

Solution

Begin by isolating the absolute value as follows:

$$3|x+1|-2=-11$$

$$3|x+1|=-11+2$$

$$3|x+1|=-9$$

$$|x+1|=-3$$

Since an absolute value will not produce non-negative values as solutions, this problem has **no solution**.

3.2 Inequalities and Intervals

Solve and express the solutions in terms of intervals whenever possible.

(a) $-\frac{1}{3}|6-5x|+2 \geq -1$

(b) $\frac{3}{|5-2x|} < 2$

(c) $\frac{x+1}{2x-3} > 2$

Solution

(a) First, isolate the absolute value:

$$\begin{aligned} -\frac{1}{3}|6-5x|+2 &\geq -1 \\ -\frac{1}{3}|6-5x| &\geq -3 \\ (\cancel{-3})\frac{1}{\cancel{-3}}|6-5x| &\leq -3(-3) \\ |6-5x| &\leq 9 \end{aligned}$$

Remember, when you multiply or divide by a negative value, the inequality sign switches directions

Next, set the absolute value to both its positive and negative solutions and solve for x:

$$\begin{aligned} 6-5x &\leq 9 \\ \cancel{6}-5x\cancel{-6} &\leq 9-6 \\ -5x &\leq 3 \\ \left(\frac{1}{\cancel{-5}}\right)\cancel{-5}x &\geq 3\left(\frac{1}{-5}\right) \\ x &\geq -\frac{3}{5} \end{aligned}$$

$$\begin{aligned} 6-5x &\geq -9 \\ \cancel{6}-5x\cancel{-6} &\geq -9-6 \\ -5x &\geq -15 \\ \left(\frac{1}{\cancel{-5}}\right)\cancel{-5}x &\leq -15\left(\frac{1}{-5}\right) \\ x &\leq 3 \end{aligned}$$

Finally, write solution on interval form:

$$\left[-\frac{3}{5}, 3\right]$$

(b) Unlike other inequalities, we begin this problem by multiplying $|5-2x|$ over to the right hand side. Since $|5-2x|$ will be a nonnegative value, this is allowed and will not change the direction of the inequality:

$$\begin{aligned} \frac{3}{|5-2x|} &< 2 \\ 3 &< 2|5-2x| \end{aligned}$$

Next, we divide by 2 in order to isolate the absolute value:

$$\begin{aligned} \frac{3}{2} &< \frac{\cancel{2}}{\cancel{2}}|5-2x| \\ \frac{3}{2} &< |5-2x| \end{aligned}$$

Next, set the absolute value to both its positive and negative solutions and solve for x:

$$\begin{aligned} \frac{3}{2} &< 5-2x \\ -2x &> \frac{3}{2}-5 \\ -2x &> -\frac{7}{2} \\ \frac{\cancel{-2}}{\cancel{-2}}x &< \left(-\frac{7}{2}\right)\left(-\frac{1}{2}\right) \\ x &< \frac{7}{4} \end{aligned} \qquad\qquad \begin{aligned} -\frac{3}{2} &> 5-2x \\ -2x &< \frac{3}{2}-5 \\ -2x &< -\frac{13}{2} \\ \frac{\cancel{-2}}{\cancel{-2}}x &> \left(-\frac{13}{2}\right)\left(-\frac{1}{2}\right) \\ x &> \frac{13}{4} \end{aligned}$$

Finally, write solution on interval form:

$$\left(-\infty, \frac{7}{4}\right) \cup \left(\frac{13}{4}, \infty\right)$$

(c) Begin by moving everything to one side of the inequality to produce a 0 on one of the sides:

$$\frac{x+1}{2x-3} - 2 > 0$$

Next, combine the terms by finding a common denominator as follows:

$$\begin{aligned} \frac{x+1}{2x-3} - \frac{2(2x-3)}{2x-3} &> 0 \\ \frac{x+1-4x+6}{2x-3} &> 0 \\ \frac{-3x+7}{2x-3} &> 0 \end{aligned}$$

We can determine the solution here using a sign chart. However, to do so we must first determine the zeros of both the denominator and numerator:

$$\begin{aligned} -3x+7 &= 0 \\ -3x &= -7 \\ x &= \frac{7}{3} \end{aligned} \qquad\qquad \begin{aligned} 2x-3 &= 0 \\ 2x &= 3 \\ x &= \frac{3}{2} \end{aligned}$$

Now we can complete the sign chart as follows:

		$\frac{3}{2}$		$\frac{7}{3}$	
$-3x+7$	+		+		-
$2x-3$	-		+		+
$\frac{-3x+7}{(2x-3)}$	-		+		-

From this, we get our answer to be

$$\left(\frac{3}{2}, \frac{7}{3}\right)$$

4 Functions and Graphs

4.1 Mid-Point

Find a general form of an equation for the perpendicular bisector of a segment AB

$$A(3,-1) \qquad B(-2,6)$$

Solution

First, we need to find the slope od the segment AB using the slope formula.

$$m = \frac{\triangle y}{\triangle x} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\Rightarrow \frac{6-(-1)}{-2-3} = -\frac{7}{5}$$

Next we need to find the slope of the perpendicular bisector. Remember:

$$\text{Perpendicular slope} = -\frac{1}{m}$$
$$\text{Parallel slope} = m$$

The slope of the perpendicular bisector is $\frac{5}{7}$. A perpendicular bisector is defined as a line segment that is both perpendicular to a side and and passes through its midpoint, for this reason we need to find the mid-point of segment AB next.

$$\text{Mid-Point} = \left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

$$\Rightarrow \left(\frac{-2+3}{2}, \frac{6-1}{2}\right)$$

$$\Rightarrow \left(\frac{1}{2}, \frac{5}{2}\right)$$

Finally, we need to put all the pieces together in order to find the equation of the line. We will begin by first setting our equation in *point − slope*.

$$(y - y_0) = m(x - x_0)$$
$$\left(y - \frac{5}{2}\right) = \frac{5}{7}\left(x - \frac{1}{2}\right)$$

Finally, rearrange into *General Form*.

$$
\begin{aligned}
Ax + By &= C \\
7\left(y - \frac{5}{2}\right) &= \frac{5}{\cancel{7}}\left(x - \frac{1}{2}\right)\cancel{7} \\
7y - \frac{35}{2} &= 5x - \frac{5}{7} \\
7y - 5x &= -\frac{5}{7} + \frac{35}{2} \\
7y - 5x &= \frac{30}{2} \\
-1\Big(7y - 5x &= 15\Big) \\
5x - 7y &= -15
\end{aligned}
$$

4.2 Circles

(a) Find an equation of the circle that satisfies the stated conditions.

End points of a diameter $A(4, -3)$ and $B(-2, 7)$

Solution

We begin by realizing that the center of any circle is the midpoint of a diameter, so the center of the circle can be determined as follows.

$$\left(\frac{4+(-2)}{2}, \frac{(-3)+7}{2}\right) = (1, 2)$$

To find the length of the diameter of the circle we use the distance formula.

$$D = \sqrt{(4-(-2))^2 + ((-3)-7)^2} = \sqrt{136} = 2\sqrt{34}$$

Then the radius of the circle is one half of the diameter, so $r = \frac{D}{2} = \sqrt{34}$. Using the general equation of a circle, we have that the equation we need is

$$(x-1)^2 + (y-2)^2 = 34$$

(b) Find the center and radius of the circle with the given equation

$$2x^2 + 2y^2 - 12x + 4y - 15 = 0$$

Solution

We begin by dividing by 2 and then simplifying further by separating the x terms and the y terms and also moving the constant to the right hand side.

$$\begin{aligned} 2x^2 + 2y^2 - 12x + 4y - 15 &= 0 \\ x^2 + y^2 - 6x + 2y - \frac{15}{2} &= 0 \\ x^2 + y^2 - 6x + 2y &= \frac{15}{2} \\ x^2 - 6x + y^2 + 2y &= \frac{15}{2} \end{aligned}$$

Then we complete the square to put the equation in the standard form of a circle.

$$\begin{aligned} (x^2 - 6x + 9) + (y^2 + 2y + 1) &= \frac{15}{2} + 9 + 1 \\ (x-3)^2 + (y+1)^2 &= \frac{35}{2} \end{aligned}$$

From here we have that the center of the circle is at $(3, -1)$ and that the radius of the circle is given by

$$r = \sqrt{\frac{35}{2}} = \frac{\sqrt{70}}{2}$$

4.3 Piecewise Functions

Find the domain and sketch the graph of

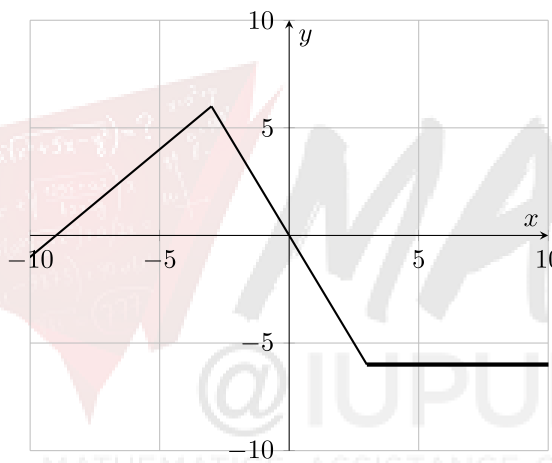
$$f(x) = \begin{cases} x+9 & \text{if } x < -3 \\ -2x & \text{if } |x| \leq 3 \\ -6 & \text{if } x > 3 \end{cases}$$

Solution

The domain of the function is all numbers x for which $f(x)$ is defined. Since we may plug any number into the function, the domain is given by

$$(-\infty, \infty)$$

To begin graphing the function, we graph it on the three intervals $(-\infty, -3)$, $[-3, 3]$ and $(3, \infty)$. Since they are all lines on each of these intervals, we arrive at the following graph.



4.4 Inequality

Solve and express the solution in terms of intervals whenever possible.

$$\frac{x-2}{x^2-3x-10} \geq 0$$

Solution

Begin by factoring the denominator.

$$\frac{x-2}{x^2-3x-10} = \frac{x-2}{(x-5)(x+2)}$$

From here, make a sign chart to determine where the function is positive or negative.

		-2		2		5	
$x-2$	-		-		+		+
$x+2$	-		+		+		+
$x-5$	-		-		-		+
$\frac{x-2}{(x+2)(x-5)}$	-		+		-		+

Thus, the solution set is:

$$(-2, 2] \cup (5, \infty)$$

4.5 Difference Quotient

Simplify the difference quotient.

$$\frac{f(x+h)-f(x)}{h} \quad \text{if} \quad h \neq 0$$

$$\text{where } f(x) = x^2 + 5$$

Solution

Begin by plugging in values x and $x+h$ for the function and then simplifying.

$$\frac{f(x+h)-f(x)}{h} = \frac{((x+h)^2+5)-(x^2+5)}{h}$$

Now expand the numerator to simplify the expression

$$\frac{(x^2+2xh+h^2+5)-(x^2+5)}{h} = \frac{2xh+h^2}{h} = 2x+h$$

4.6 Graphs of Functions

Determine whether f is even, odd, or neither even nor odd.

$$f(x) = 8x^3 - 3x^2$$

Solution

To determine whether the function is odd or even, recall that a function is even if $f(-x) = f(x)$ and a function is odd if $f(-x) = -f(x)$. Check to see if either of these equalities holds.

$$\begin{aligned} f(x) &= 8^3 - 3x^2 \\ f(-x) &= 8(-x)^3 - 3(-x)^2 = -8x^3 - 3x^2 \\ -f(x) &= -(8x^2 - 3x^2) = -8x^2 + 3x^2 \end{aligned}$$

We can see from here that $f(-x) \neq f(x)$ and $f(x) \neq -f(x)$, so the function is neither even nor odd.

4.7 Parabola

Express $f(x)$ in the form $a(x-h)^2+k$ and graph.

$$f(x) = -3x^2 - 6x - 5$$

Solution

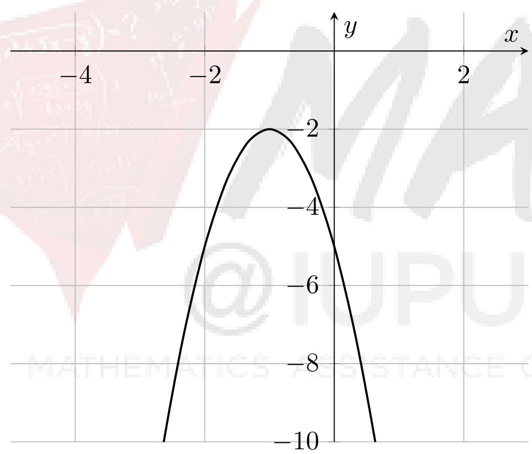
Begin by completing the square.

$$\begin{aligned} f(x) &= -3x^2 - 6x - 5 \\ &= -3(x^2 + 2x) - 5 \\ &= -3(x^2 + 2x + 1) - 5 + 3 \\ f(x) &= -3(x+1)^2 - 2 \end{aligned}$$

To begin graphing, let $f(x) = y$ and notice the equation is the equation of a parabola.

$$\begin{aligned} y &= -3(x+1)^2 - 2 \\ (y+2) &= -3(x+1)^2 \end{aligned}$$

From here, it can be seen that the parabola opens downwards with vertex at $(-1, -2)$.



4.8 Composite Functions

For $f(x)$ and $g(x)$ below find

(a) $(f \circ g)$ and its domain.

(b) $(g \circ f)$ and its domain.

$$f(x) = \sqrt{3-x} \qquad g(x) = \sqrt{x^2-16}$$

Solution

(a) Begin by plugging in the function $g(x)$ into $f(x)$.

$$(f \circ g)(x) = f(g(x)) = \sqrt{3-g(x)} = \sqrt{3-\sqrt{x^2-16}}$$

The domain of $(f \circ g)$ is everywhere that the inside of both square roots is non-negative, since we cannot take the square root of a negative number. So we have

$$\begin{aligned} 3-\sqrt{x^2-16} &\geq 0 \\ 3 &\geq \sqrt{x^2-16} \\ 9 &\geq x^2-16 \\ 25 &\geq x^2 \\ 5 &\geq |x| \\ -5 \leq x &\leq 5 \end{aligned} \qquad \begin{aligned} x^2-16 &\geq 0 \\ x^2 &\geq 16 \\ |x| &\geq 4 \\ x \geq 4 \text{ or } x &\leq -4 \end{aligned}$$

Combining these results in interval notation gives the domain to be $[-5,-4]\bigcup[4,5]$.

(b) Begin by plugging in the function $f(x)$ into $g(x)$.

$$(g \circ f)(x) = g(f(x)) = \sqrt{(f(x))^2-16} = \sqrt{(\sqrt{3-x})^2-16} = \sqrt{(3-x)-16} = \sqrt{-x-13}$$

To find the domain, set the inside of the square root to be greater than or equal to zero as above.

$$\begin{aligned} -x-13 &\geq 0 \\ -13 &\geq x \end{aligned}$$

So the domain, in interval notation, is $(-\infty,-13]$.

4.9 Polynomial Functions of Degree Greater than 2

Find all values of x such that $f(x) > 0$ and all x such that $f(x) < 0$, and sketch the graph of f.

$$f(x) = x^3 + 2x^2 - 4x - 8$$

Solution

To find where the function is positive, begin by factoring the function as follows.

$$\begin{aligned} f(x) &= x^3 + 2x^2 - 4x - 8 \\ &= x^2(x+2) - 4(x+2) \\ &= (x^2 - 4)(x+2) \\ &= (x-2)(x+2)(x+2) \\ &= (x+2)^2(x-2) \end{aligned}$$

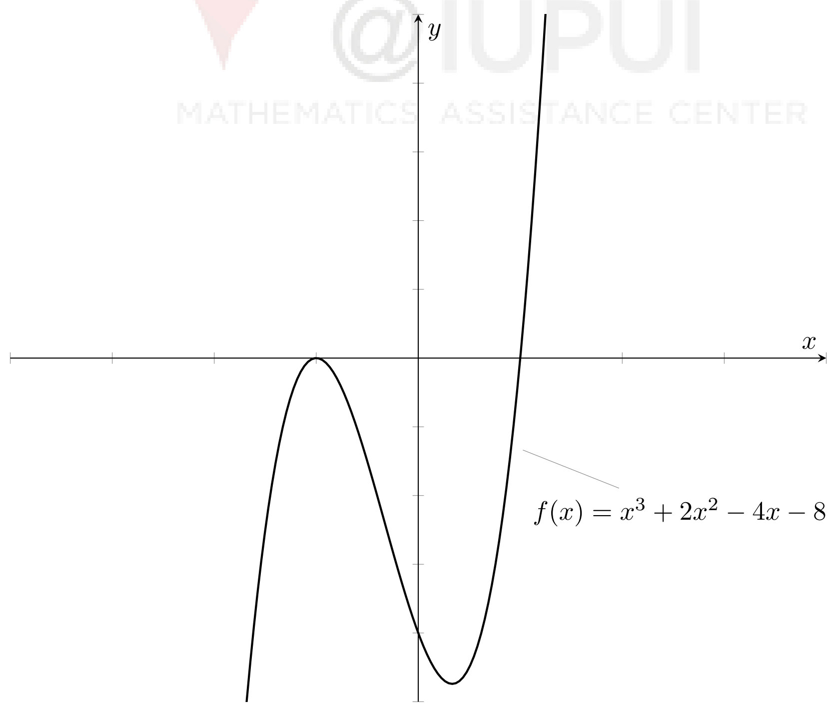
Since $(x+2)^2$ is a perfect square, it is always positive or zero. Because of this, the function is positive only when $(x-2) > 0$, or in other words, when $x > 2$. In interval notation, $f(x) > 0$ when x is in the interval

$$(2, \infty)$$

To find where the function is negative, use the factorization above. It is easy to see that $f(x) < 0$ only when $(x-2) < 0$ and $x \neq -2$, since the term $(x+2)^2$ is always positive or equal to zero. Then in interval notation, $f(x) < 0$ when x is in

$$(-\infty, -2) \bigcup (-2, 2)$$

To graph the function, use the fact that $f(x)$ is negative in $(-\infty, -2)$ and $(-2, 2)$ and $f(x)$ is positive in $(2, \infty)$.



5 Properties of Division

5.1 Long Division

Find the quotient $q(x)$ and remainder $r(x)$ if $f(x)$ is divided by $p(x)$.

$$f(x) = 3x^3 + 2x - 4 \qquad\qquad p(x) = 2x^2 + 1$$

Solution

Use long division in the normal way to get

$$\begin{array}{r|l} & \frac{3}{2}x \\ \hline 2x^2+1 & 3x^3 + 2x - 4 \\ & -\,3x^3 - \frac{3}{2}x \\ \cline{2-2} & \frac{1}{2}x \end{array}$$

Hence the quotient and remainder are

$$q(x) = \frac{3}{2}x$$
$$r(x) = \frac{1}{2}x - 4$$

5.2 Synthetic Division

Use synthetic division to find the quotient and remainder if $f(x)$ is divided by $p(x)$

$$f(x) = 2x^3 - 3x^2 + 4x - 5 \qquad\qquad p(x) = x - 2$$

Solution

Use the standard synthetic division procedure to obtain

$$\begin{array}{r|rrrr} & 2 & -3 & 4 & -5 \\ 2 & & 4 & 2 & 12 \\ \hline & 2 & 1 & 6 & 7 \end{array}$$

$$q(x) = 2x^2 + x + 6$$
$$r(x) = 7$$

6 Inverse Functions

6.1 Finding Inverse

Find the inverse function of f.

$$f(x) = \frac{3x+2}{2x-5}$$

Solution

Begin by solving for x in terms of $f(x)$.

$$\begin{aligned} f(x) &= \frac{3x+2}{2x-5} \\ (2x-5)f(x) &= 3x+2 \\ 2xf(x) - 5f(x) &= 3x+2 \\ 2xf(x) - 3x &= 5f(x)+2 \\ x(2f(x)-3) &= 5f(x)+2 \\ x &= \frac{5f(x)+2}{2f(x)-3} \end{aligned}$$

From here, we replace x with $f^{-1}(x)$ and $f(x)$ with x to get our answer.

$$f^{-1}(x) = \frac{5x+2}{2x-3}$$

6.2 Domain and Range of f^{-1}

Determine the domain and range of f^{-1} for the given function.

$$f(x) = -\frac{4x+5}{3x-8}$$

Solution

Since the domain of $f(x)$ is $\left(-\infty, \frac{8}{3}\right) \bigcup \left(\frac{8}{3}, \infty\right)$, we have that the range of $f^{-1}(x)$ is simply

$$\left(-\infty, \frac{8}{3}\right) \bigcup \left(\frac{8}{3}, \infty\right)$$

To find the domain of $f^{-1}(x)$, we first need to find $f^{-1}(x)$

$$\begin{aligned} y &= -\frac{4x+5}{3x-8} \\ x &= -\frac{4y+5}{3y-8} \\ 3yx - 8x &= -4y + 5 \\ y(3x+4) &= 8x + 5 \\ \frac{y\cancel{(3x+4)}}{\cancel{3x+4}} &= \frac{8x+5}{3x+4} \\ y &= \frac{8x+5}{3x+4} \end{aligned}$$

The domain of $f^{-1}(x)$ is $\left(-\infty, -\frac{4}{3}\right) \bigcup \left(-\frac{4}{3}, \infty\right)$.

7 Exponential and Logarithmic Functions

7.1 Exponential Functions

Solve the equation

(a) $3^{x+4} = 2^{1-3x}$

(b) $2^{2x-3} = 5^{x-2}$

Solution

(a)

$$\begin{aligned}
3^{x+4} &= 2^{1-3x} \\
\log(3^{x+4}) &= \log(2^{1-3x}) \\
(x+4)\log(3) &= (1-3x)\log(2) \\
x\log(3) + 4\log(3) &= \log(2) - 3x\log(2) \\
x\log(3) + 3x\log(2) &= \log(2) - \log(3^4) \\
x\big[\log(3) + \log(2^3)\big] &= \log\left(\frac{2}{81}\right) \\
x &= \frac{\log\left(\frac{2}{81}\right)}{\log(3\times 8)} \\
x &= \frac{\log\left(\frac{2}{81}\right)}{\log(24)} \\
x &\approx -1.16
\end{aligned}$$

(b)

$$\begin{aligned}
2^{2x-3} &= 5^{x-2} \\
(2x-3)\log(2) &= (x-2)\log(5) \\
2x\log(2) - 3\log(2) &= x\log(5) - 2\log(5) \\
x\log(2^2) - x\log(5) &= \log(2^3) - \log(5^2) \\
x\big[\log(4) - \log(5)\big] &= \log\left(\frac{8}{25}\right) \\
x &= \frac{\log\left(\frac{8}{25}\right)}{\log\left(\frac{4}{5}\right)} \\
x &\approx 5.11
\end{aligned}$$

7.2 Compound Interest Formula

If \$1000 is invested at a rate of 12% per year compounded monthly, find the amount after:

(a) 1 month

(b) 6 months

(c) 1 year

(d) 20 years

Solution

(a) Here, we need to use the following formula for calculating payments:

$$P = P_0\left(1+\frac{r}{n}\right)^{nt}$$

where t is the number of periods that have passed (i.e. how many times interest has been compounded) and n is the number of times interest is compounded annually. In this case, we need to convert the months to years as it interest is compounded annually. Thus:

$$P = 1000\left(1+\frac{0.12}{12}\right)^{\frac{1}{12}*12} = \$1010$$

(b) Similar to the previous part:

$$\begin{aligned} P &= P_0\left(1+\frac{r}{n}\right)^{nt} \\ &= 1000\left(1+\frac{0.12}{12}\right)^{6} \\ &= \$1061.52 \end{aligned}$$

(c) Here, we need to consider the number of months there are in a year rather than simply plugging in the number of years:

$$\begin{aligned} P &= 1000\left(1+\frac{0.12}{12}\right)^{12} \\ &= \$1126.83 \end{aligned}$$

(d) 20 years $= 20 \times 12 = 240$ months. Therefore:

$$\begin{aligned} P &= 1000\left(1+\frac{0.12}{12}\right)^{240} \\ &= \$10892.55 \end{aligned}$$

7.3 Continuously Compounded Interest Formula

If $P = 1000$ dollars is deposited in a savings account that pays interest at a rate of $r = 8\frac{1}{4}\%$ per year compounded continuously, find the balance after $t = 5$ years.

Solution

Here,we will use the formula for interested compounded continuously:

$$P = P_0 e^{rt}$$

Where t is the number of years. First, we will write the rate as a fraction and then plug it into the formula:

$$r = 8\frac{1}{4} = \frac{8 \times 4 + 1}{4} = \frac{33}{4}\%$$
$$r = \frac{33}{4} \times \frac{1}{100} = \frac{33}{400}$$

Thus:

$$P = (1000)e^{(33/400)(5)}$$
$$= \$1,510.59$$

7.4 Natural Exponetial Function

Find the zeros of f.

$$f(x) = x^3(4e^{4x}) + 3x^2e^{4x}$$

Solution

First, let us set up our equation:

$$x^3(4e^{4x}) + 3x^2e^{4x} = 0$$

now, since e^{4x} is always positive and never zero, we can take it out as a common factor and divide by it:

$$e^{4x}(4x^3 + 3x^2) = 0$$
$$4x^3 + 3x^2 = \frac{0}{e^{4x}}$$
$$4x^3 + 3x^2 = 0$$

Now it is simple to factor the equation at hand and find the zeros:

$$x^2(4x + 3) = 0$$
$$x = 0 \quad \text{and} \quad 4x + 3 = 0 \implies x = -3/4$$

The population $N(t)$ (in millions) of the United States t years after 1980 may be approximated by the formula $N(t) = 227e^{0.007t}$.

(a) When will the population be twice what it was in 1980?

Solution

We can see that the population in 1980 was 227 million (you can see this by simply plugging in $t = 0$). Now, we would like to find the time at which $N(t) = 454$:

$$\begin{aligned} 454 &= 227e^{0.007t} \\ 2 &= e^{0.007t} \\ \ln(2) &= 0.007t \\ t &= \frac{\ln(2)}{0.007} \approx 99 \end{aligned}$$

Thus, the population will be twice what it was in 1980 in the year $1980 + 99 =$ **2079**.

Use natural logarithms to solve for x in terms of y.

$$y = \frac{e^x - e^{-x}}{2}$$

Solution

We can begin by writing e^{-x} as follows:

$$y = \frac{e^x - \frac{1}{e^x}}{2}$$

From here, we attempt to isolate the term e^x:

$$\begin{aligned} 2y &= e^x - \frac{1}{e^x} \\ 2ye^x &= e^{2x} - 1 \\ 0 &= e^{2x} - 2ye^x - 1 \end{aligned}$$

Now using the quadratic formula:

$$\begin{aligned} e^x &= \frac{2y \pm \sqrt{4y^2+4}}{2} \\ e^x &= y \pm \sqrt{y^2+1} \end{aligned}$$

Here, notice that e^x is always positive, so we can use the positive answer. However, notice that the expression $y - \sqrt{y^2+1}$ is always negative since $y - \sqrt{y^2+1} < 0 \implies y^2 < y^2 + 1 \implies 0 < 1$ which is obviously true. Thus:

$$\begin{aligned} e^x &= y + \sqrt{y^2+1} \\ x &= \ln(y + \sqrt{y^2+1}) \end{aligned}$$

7.5 Properties of Logarithms

Solve for t using logarithms with base a

$$A = Ba^{Ct} + D$$

Solution

To do this, we simply isolate the term with t in it as follows:

$$a^{Ct} = \frac{A - D}{B}$$

Now we can take the natural logarithm of base a on both sides:

$$Ct = \log_a\left(\frac{A - D}{B}\right)$$
$$t = \frac{1}{C}\log_a\left(\frac{A - D}{B}\right)$$

Solve the equation

$$\ln(-4-x)+\ln 3=\ln(2-x)$$

Solution

we will begin by simplifying the problem using the properties of logarithms:

$$\begin{aligned}\ln(-4-x)+\ln 3&=\ln(2-x)\\ \ln(3(-4-x))&=\ln(2-x)\end{aligned}$$

Now, we can exponentiate both sides of the equation to remove the logarithm:

$$\begin{aligned}e^{\ln(3(-4-x))}&=e^{\ln(2-x)}\\ 3(-4-x)&=2-x\\ -3x-12&=2-x\\ -14&=2x\\ x&=-7\end{aligned}$$

Since we exponentiated both sides of the equation, we must check our answers for extraneous solutions by plugging it back in:

$$\begin{aligned}\ln(-4-(-7))+\ln(3)&\stackrel{?}{=}\ln(2-(-7))\\ \ln(3)+\ln(3)&\stackrel{?}{=}\ln(9)\\ \ln(3\times 3)&=\ln(9)\end{aligned}$$

Thus, the result checks off and indeed, $x=-7$ is a solution.

Solve the equation

$$\log_3(x+3) + \log_3(x+5) = 1$$

Solution

We will use the properties of logarithms to simplify the equation as follows:

$$\log_3(x+3) + \log_3(x+5) = 1$$
$$\log_3((x+3)(x+5)) = 1$$

Here, we exponentiate both sides using a base of 3 to remove the logarithm:

$$3^{\log_3((x+3)(x+5))} = 3^1$$
$$(x+3)(x+5) = 3$$
$$x^2 + 8x + 15 = 3$$
$$x^2 + 8x + 12 = 0$$

From here, we can either use the quadratic formula or factor the expression on the left hand side of the equation (we choose to factor here, but the quadratic formula will give the same solutions):

$$(x+2)(x+6) = 0$$

This means that either $x = -6$ or $x = -2$. We must check our answer since we exponentiated both sides of the equation in the process.

For $x = -2$:

$$\log_3(-2+3) + \log_3(-2+5) \stackrel{?}{=} 1$$
$$\log_3(1) + \log_3(3) \stackrel{?}{=} 1$$
$$0 + 1 = 1$$

So $x = -2$ is a valid solution.

For $x = -6$:

$$\log_3(-6+3) + \log_3(-6+5) \stackrel{?}{=} 1$$
$$\log_3(-3) + \log_3(-1) \stackrel{?}{=} 1$$

Notice that the logarithm is not defined for negative values, thus $x = -6$ is not a valid solution. The only solution here is $x = -2$.

Find the exact solution, using common logarithms, and a two-decimal-place approximation, when appropriate.

$$\log(x-4) - \log(3x-10) = \log\left(\frac{1}{x}\right)$$

Solution

Once again, we will use the same simplification step to rewrite the equation as follows:

$$\log(x-4) - \log(3x-10) = \log\left(\frac{1}{x}\right)$$
$$\log\left(\frac{x-4}{3x-10}\right) = \log\left(\frac{1}{x}\right)$$

Now, exponentiating both sides with a base 10:

$$10^{\log\left(\frac{x-4}{3x-10}\right)} = 10^{\log\left(\frac{1}{x}\right)}$$
$$\frac{x-4}{3x-10} = \frac{1}{x}$$

We can now cross multiply and proceed to solve the equation at hand:

$$x(x-4) = 3x-10$$
$$x^2 - 4x = 3x - 10$$
$$x^2 - 7x + 10 = 0$$

Once again, either the quadratic formula or factoring will work (we will use the quadratic formula this time around):

$$x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4(1)(10)}}{2}$$
$$x = \frac{7 \pm 3}{2}$$

Hence, either $x = 5$ or $x = 2$. We must check both answers since we exponentiated both sides of the equation in the process.

For $x = 5$:

$$\log(5-4) - \log(3(5)-10) \stackrel{?}{=} \log\left(\frac{1}{5}\right)$$
$$\log(1) - \log(5) \stackrel{?}{=} -\log(5)$$
$$0 - \log(5) = -\log(5)$$

Thus, $x = 5$ is a valid solution.

For $x = 2$:

$$\log(2-4) - \log(3(2)-10) \stackrel{?}{=} \log\left(\frac{1}{2}\right)$$

This results in negative arguments in the logarithm function, which is not defined. Thus, $x = 2$ is not a valid solution, and the only solution is $x = 5$.

Find the exact solution, using common logarithms, and a two-decimal-place approximation, when appropriate.

$$4^x - 3(4^{-x}) = 8$$

Solution

We will begin by rewriting 4^{-x} as follows:

$$4^x - 3(4^{-x}) = 8$$
$$4^x - \frac{3}{4^x} = 8$$

Multiplying both sides by 4^x clears out the denominator:

$$(4^x)^2 - 3 = 8(4^x)$$
$$(4^x)^2 - 8(4^x) - 3 = 0$$

Notice that this is a quadratic equation, which means we can solve this by first making the substitution $u = 4^x$ and using the quadratic formula:

$$u^2 - 8u - 3 = 0$$
$$u = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(-3)}}{2}$$
$$u = \frac{8 \pm \sqrt{76}}{2}$$
$$u == \frac{8 \pm \sqrt{4 \times 19}}{2}$$
$$u = \frac{8 \pm 2\sqrt{19}}{2}$$
$$u = 4 \pm \sqrt{19}$$

Now we must substitute back the expression $u = 4^x$ to attain the solutions in terms of x:

$$4^x = 4 + \sqrt{19} \quad \text{or} \quad 4^x = 4 - \sqrt{19}$$
$$x = \log_4(4 + \sqrt{19}) \quad \text{or} \quad x = \log_4(4 - \sqrt{19})$$

Notice that $4 - \sqrt{19}$ is a negative value, and thus we cannot apply the logarithm to it. thus, the only solution is:

$$x = \log_4(4 + \sqrt{19}) \approx 1.53$$